

# Inequalities for regular near polygons and hemisystem-like subgraphs with classical parameters

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July 13 2010, AGC 2010

## Outline

- Introduction
- Inequality (and the case of equality)
- Construction of distance-regular graph as subgraph

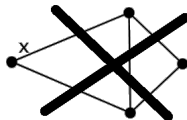
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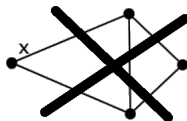
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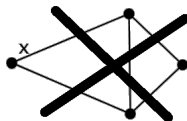


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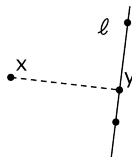
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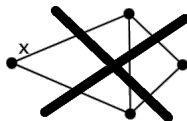
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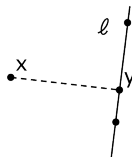


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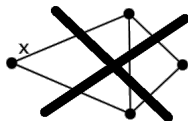
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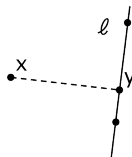
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Simplest example of near  $2d$ -gon: ordinary  $2d$ -gons!

In any graph  $\Gamma$ :  $\Gamma_i(x)$  denotes vertices at distance  $i$  from  $x$ ,  
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- $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$
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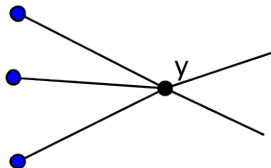
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Hence the valency is  $k = s(t + 1)$ .



$\Gamma$  is regular near  $2d$ -gon of order  $(s, t)$ :  $s + 1$  points on singular line.  
If  $d(x, y) = j$  then through  $y$ :

- $c_j$  singular lines with 1 point at distance  $j - 1$  from  $x$ ,  
and its  $s$  other points at distance  $j$  from  $x$ ,
- $(t + 1) - c_j$  singular lines with  $y$  at distance  $j$  from  $x$ ,  
and its  $s$  other points at distance  $j + 1$  from  $x$ .

## Examples of regular near $2d$ -gons: dual polar graphs

Consider vector space  $V$  and non-degenerate sesquilinear/quadratic form  $f$ .  
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$$c_j = (t_2^j - 1)/(t_2 - 1)$$

		$(s, t_2)$
$Q^+(2d - 1, q)$	$D_d(q)$	$(1, q)$
$H(2d - 1, q^2)$	${}^2A_{2d-1}(q)$	$(q, q^2)$
$Q(2d, q)$	$B_d(q)$	$(q, q)$
$W(2d - 1, q)$	$C_d(q)$	$(q, q)$
$H(2d, q^2)$	${}^2A_{2d}(q)$	$(q^3, q^2)$
$Q^-(2d + 1, q)$	${}^2D_{d+1}(q)$	$(q^2, q)$

## The original Higman inequality for $d = 2$ (1974)

In a regular near  $2d$ -gon with  $d = 2$  (or *generalized quadrangle*) of order  $(s, t)$  with  $s > 1$ :

$$c_2 = t + 1 \leq s^2 + 1,$$

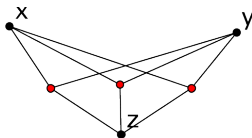
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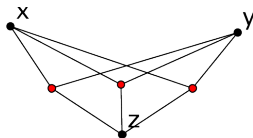
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We want similar inequalities and *triple intersection numbers* for  $d > 2!$

For any distance-regular graph  $\Gamma$  with vertex set  $\Omega$  and diameter  $d$ :  
Define  $(|\Omega| \times |\Omega|)$ -matrices  $A_0, \dots, A_d$ :

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & \text{if } d(x,y) \neq i \end{cases} .$$

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*Eigenvalues of  $\Gamma$* : the eigenvalues of  $A_1$ .

$\mathbb{R}^\Omega$  decomposes orthogonally into  $d + 1$  real eigenspaces:

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Every subset  $S \subseteq \Omega$  has *characteristic vector*  $\chi_S \in \mathbb{R}^\Omega$ :

$(\chi_S)_x = 1$  if  $x \in S$ ,  $(\chi_S)_x = 0$  if  $x \notin S$ .

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## Applying Delsarte's linear programming bound

For every  $v \in \mathbb{R}^\Omega$ :

$$v^T M v \geq 0,$$

with equality iff  $Mv = 0$  or equivalently  $v \in (V_d)^\perp$ .

## Theorem

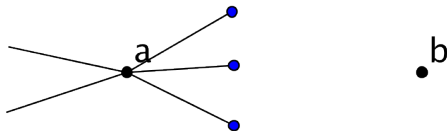
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If equality holds, then if  $d(a, b) = j$ :

$$M\left(s \frac{s^{2j-2} - 1}{s^2 - 1} \chi_{\{a\}} + (-1)^j s^{j-1} \chi_{\{b\}} + \chi_T\right) = 0.$$



with  $T$  the  $c_j$  points in  $\Gamma_1(a) \cap \Gamma_{j-1}(b)$ .

## Proof

$\Gamma$  is regular near  $2d$ -gon of order  $(s, t)$  ( $s > 1$ ).

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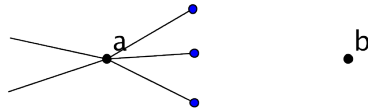
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Note that for any two subsets  $S_1, S_2$ :

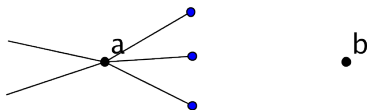
$(\chi_{S_1})^T A_i \chi_{S_2} =$  number of pairs  $(x_1, x_2)$  in  $(S_1 \times S_2)$  with  $d(x_1, x_2) = i$ .

## Proof (continued)



Note: 2 distinct points in  $T = \Gamma_1(a) \cap \Gamma_{j-1}(b)$  must be at distance 2.

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So with

$$v = \alpha \chi_{\{a\}} + \beta \chi_{\{b\}} + \gamma \chi_T, M = A_0 - \frac{A_1}{s} + \frac{A_2}{s^2} - \dots + \frac{(-1)^d A_d}{s^d}.$$

we obtain that  $v^T M v$  is  $((\alpha, \beta, \gamma)^T F(\alpha, \beta, \gamma))/s^j$  with:

$$F = \begin{pmatrix} s^j & (-1)^j & -s^{j-1}c_j \\ (-1)^j & s^j & (-1)^{j-1}sc_j \\ -s^{j-1}c_j & (-1)^{j-1}sc_j & c_j s^{j-2}(s^2 + c_j - 1) \end{pmatrix}.$$

## Proof (last part)

$M$  is projection up to positive scalar, so  $v^T M v \geq 0$  and thus  $(\alpha, \beta, \gamma)^T F(\alpha, \beta, \gamma) \geq 0, \forall \alpha, \beta, \gamma \in \mathbb{R}$ .

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$$\text{Det}(F) = c_j s^{j-2} (s^2 - 1) ((s^{2j} - 1) - c_j (s^2 - 1)) \geq 0 \iff c_j \leq \frac{s^{2j} - 1}{s^2 - 1}.$$

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$Mv = 0 \iff v^T M v = 0$  if and only if  $c_j = \frac{s^{2j} - 1}{s^2 - 1}$  and  $(\alpha, \beta, \gamma)$  is scalar multiple of:

$$\left( s \frac{s^{2j-2} - 1}{s^2 - 1}, (-1)^j s^{j-1}, 1 \right).$$



Triple intersection numbers if SOME  $c_j = \frac{s^{2j}-1}{s^2-1}$

- We know that if  $d(a, b) = j$  and  $(\alpha, \beta, \gamma) = \left( s \frac{s^{2j}-1}{s^2-1}, (-1)^j s^{j-1}, 1 \right)$ :

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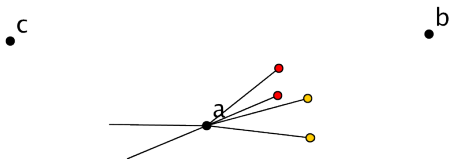
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That identity allows us to compute how many of each!

## Triple intersection numbers if SOME $c_j = \frac{s^{2j}-1}{s^2-1}$ (continued)



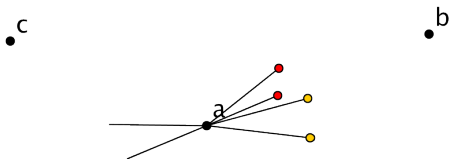
We assume  $d(a, b) = j, d(a, c) = d, d(b, c) = k$ .

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The case  $d = j = k = 2$  from Introduction

If  $c_2 = (s^4 - 1)/(s^2 - 1)$ , we again find:  $|\Gamma_1(a) \cap \Gamma_1(b) \cap \Gamma_1(c)| = s + 1$ .

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The case  $d \geq 3$

- The only regular near  $2d$ -gon  $\Gamma$  with such parameters is the unitary dual polar graph  ${}^2A_{2d-1}(q)$ .

Distance-regular last subconstituent in near  $2d$ -gon if ALL  $c_j = \frac{s^{2j}-1}{s^2-1}$

Triple intersection numbers then yield:

subgraph  $\Gamma_d(a)$  is also distance-regular for every point  $a$ .

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  - An *m*-ovoid is a subset of points  $S$  with  $m$  points of  $S$  on each singular line.
  - If  $S$  is *m*-ovoid:
    - $a \in S$  has  $(t + 1)(m - 1)$  neighbours in  $S$ ,
    - $a \notin S$  has  $(t + 1)(m)$  neighbours in  $S$ .
- Hence  $S$  yields *equitable partition* of  $\Gamma$  into 2 parts.

## Algebraic characterization of *m*-ovoids

- We know: if  $\Gamma$  is regular near  $2d$ -gon of order  $(s, t)$  and vertex set  $\Omega$ :

$$\mathbb{R}^{\Omega} = V_0 \perp \dots \perp V_d,$$

is decomposition into eigenspaces of  $\Gamma$  with:

$V_0 = \langle \chi_{\Omega} \rangle$ : eigenspace of largest eigenvalue  $k = s(t + 1)$ ,

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Restricting  $m$  for  $m$ -ovoids  $S$  if some  $c_j = (s^{2j} - 1)/(s^2 - 1), 2 \leq j \leq d$

Pick  $a$  and  $b$  in  $S$  with  $d(a, b) = j$ .

How many of the  $c_j$  in  $T = \Gamma_1(a) \cap \Gamma_{j-1}(b)$  are also in  $S$ ?

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Working out  $\langle v, \chi_S \rangle$  yields  $|\Gamma_1(a) \cap \Gamma_{j-1}(b) \cap S|$ .
- Counting something in two ways then yields:  
 $m = (s + 1)/2$  or  $m = (s + 1)$  (trivial full set).

*m*-ovals if  $d = 2$  (so in generalized quadrangles of order  $(s, s^2)$ )

- Note: *of order*  $(s, t) = (s, s^2)$  means:  $c_2 = (s^4 - 1)/(s^2 - 1)$ .
- Restriction that  $m = (s + 1)/2$  obtained by Segre (1965), Bruen-Hirschfeld (1978), Thas (1989).
- $((s + 1)/2)$ -oval is called a *hemisystem* here.

## Induced subgraph on $((s + 1)/2)$ -ovoid

- If  $\Gamma$  is regular near  $2d$ -gon of order  $(s, t)$  with  $s > 1$  and ALL  $c_j = \frac{s^{2j}-1}{s^2-1}$ , then  $((s + 1)/2)$ -ovoid is distance-regular subgraph with:

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- So here we find classical parameters  $(d, -s, -(s + 1)/2, -((-s)^d + 1)/2)$ .

## The case $d = 2$ : generalized quadrangles of order $(s, s^2)$

- $((s + 1)/2)$ -ovoid (or hemisystem) yields strongly regular graph:  
 $srg(v, k, \lambda, \mu) =$   
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For any  $d \geq 2$ , subgraph is triangle-free for  $s = 3$ .

Meaning of those parameters  $(d, -q, -(q+1)/2, -((-q)^d + 1)/2)$

Weng (1998):

If  $\Gamma$  has classical parameters  $(d, b, \alpha, \beta)$  with  $b < -1$   
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No examples of last type for  $d = 3$  or  $d \geq 4$  seem known, but  $((q+1)/2)$ -ovoid of  ${}^2A_{2d-1}(q)$  would yield it.

Can we actually find these  $((q + 1)/2)$ -ovoids in  ${}^2A_{2d-1}(q)$ ?

Remember about dual polar graph:

- isotropic  $d$ - and  $(d - 1)$ -spaces  
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This fits into Delsarte's theory of  $t$ -designs in *regular semilattices*!

See Stanton (1986) and Munemasa (1986) for overview.

*t*-designs  $S$  of  $k$ -sets in set of size  $n$

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- one sporadic 3-design by Braun-Kerber-Laue (2003): in  $V(8, 2)$ .

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- But we want  $m$ -ovals, so  $(d - 1)$ -designs!

Non-trivial  $(d - 1)$ -designs  $S$  of isotropic  $d$ -spaces in  ${}^2A_{2d-1}(q)$  for odd  $q$ ?

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- 1-designs of right size (half) exist for  $d = 3$  and all odd  $q$   
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- Existence in  ${}^2A_{2d-1}(q)$  also implies existence in  ${}^2A_{2(d-1)-1}(q)$ ,  
so  $d = 3$  is next place to look.
- 1-designs of right size (half) exist for  $d = 3$  and all odd  $q$   
(Cossidente-Penttila (2009)).
- Conjecture by Lu-Pan-Weng (2008): *triangle-free Gewirtz graph does NOT grow*, so no such  $(d - 1)$ -designs for  $d \geq 3$  and  $q = 3$ ?

# Thank you for your attention!

(Slides (and more) on <http://cage.ugent.be/~fvanhove>)