Inequalities for regular near polygons and hemisystem-like subgraphs with classical parameters

Frédéric Vanhove
Department of Mathematics,Ghent University
fvanhove@cage.ugent.be,
Slides (and more) on `http://cage.ugent.be/~fvanhove`

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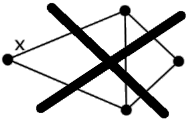
Outline

- Introduction
- Inequality (and the case of equality)
- Construction of distance-regular graph as subgraph

Near $2d$-gons

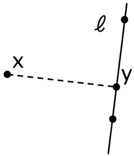
A graph Γ of diameter d is a *near* $2d$*-gon* if:

1. if x is adjacent to 2 vertices of clique of size 3 (not containing x): $\Longrightarrow$ x is adjacent to all 3.



So 2 adjacent vertices are in unique maximal clique or *singular line*.

2. $\forall$ vertex x and $\forall$ singular line ℓ, there is a unique vertex $y \in \ell$ at minimal distance from x in the graph.



(Vertices should be seen as *points*.)

Simplest example of near $2d$-gon: ordinary $2d$-gons!

In any graph Γ: $\Gamma_i(x)$ denotes vertices at distance i from x,
and $d(x, y)$ denotes distance between x and y.

A graph Γ is *distance-regular* if $\forall x, y$ with $d(x, y) = i$:

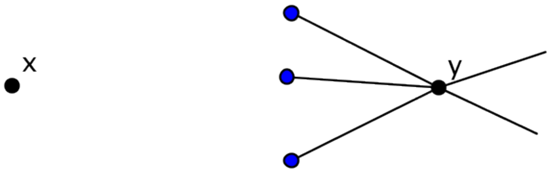
- $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$
- $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$

only depends on that distance $d(x, y) = i$.

Regular near $2d$*-gon of order* (s, t): distance-regular near $2d$-gon with:

- $s + 1$ points on each singular line,
- $t + 1$ singular lines through each point.

Hence the valency is $k = s(t + 1)$.



Γ is regular near 2d-gon of order (s, t): $s + 1$ points on singular line.
If $d(x, y) = j$ then through y:

- c_j singular lines with 1 point at distance $j - 1$ from x, and its s other points at distance j from x,
- $(t + 1) - c_j$ singular lines with y at distance j from x, and its s other points at distance $j + 1$ from x.

Examples of regular near $2d$-gons: dual polar graphs

Consider vector space V and non-degenerate sesquilinear/quadratic form f.
Polar space: all *isotropic subspaces* in V (i.e. on which f vanishes).

Dual polar graph (incidence is just symmetrized inclusion):

- points: (maximal) isotropic d-dim. subspaces (or simply *d-spaces*),
- singular lines: isotropic $(d-1)$-spaces.

$c_j = (t_2^j - 1)/(t_2 - 1)$

		(s, t_2)
$Q^+(2d-1, q)$	$D_d(q)$	$(1, q)$
$H(2d-1, q^2)$	${}^2A_{2d-1}(q)$	(q, q^2)
$Q(2d, q)$	$B_d(q)$	(q, q)
$W(2d-1, q)$	$C_d(q)$	(q, q)
$H(2d, q^2)$	${}^2A_{2d}(q)$	(q^3, q^2)
$Q^-(2d+1, q)$	${}^2D_{d+1}(q)$	(q^2, q)

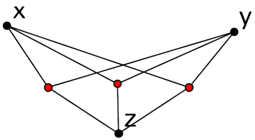
The original Higman inequality for $d = 2$ (1974)

In a regular near $2d$-gon with $d = 2$ (or *generalized quadrangle*) of order (s, t) with $s > 1$:

$$c_2 = t + 1 \leq s^2 + 1,$$

In case of equality: for every triple of non-adjacent vertices (x, y, z):

$$|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_1(z)| = s + 1.$$



We want similar inequalities and *triple intersection numbers* for $d > 2$!

For any distance-regular graph Γ with vertex set Ω and diameter d:
Define $(|\Omega| \times |\Omega|)$-matrices $A_0, \ldots, A_d$:

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & \text{if } d(x,y) \neq i \end{cases} .$$

Eigenvalues of Γ: the eigenvalues of A_1.
$\mathbb{R}^\Omega$ decomposes orthogonally into $d+1$ real eigenspaces:

$$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d.$$

Every subset $S \subseteq \Omega$ has *characteristic vector* $\chi_S \in \mathbb{R}^\Omega$:
$(\chi_S)_x = 1$ if $x \in S$, $(\chi_S)_x = 0$ if $x \notin S$.

2 specific eigenvalues of regular near 2d-gon of order (s, t)

- largest: valency $k = s(t+1)$ with eigenspace: $V_0 = \langle \chi_\Omega \rangle$,
- smallest: $-(t+1)$,

$$M = A_0 - \frac{A_1}{s} + \frac{A_2}{s^2} - \ldots + \frac{(-1)^d A_d}{s^d}$$

is (up to positive scalar) orthogonal projection from $\mathbb{R}^\Omega$ onto eigenspace V_d of $-(t+1)$.

Applying Delsarte's linear programming bound

For every $v \in \mathbb{R}^\Omega$:

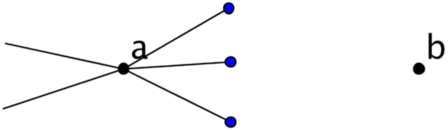
$$v^T M v \geq 0,$$

with equality iff $Mv = 0$ or equivalently $v \in (V_d)^\perp$.

Theorem

In regular near $2d$-gon of order (s, t) with $s > 1$: $c_j \leq \frac{s^{2j}-1}{s^2-1}$.
If equality holds, then if $d(a, b) = j$:

$$M\left(s\frac{s^{2j-2}-1}{s^2-1}\chi_{\{a\}} + (-1)^j s^{j-1}\chi_{\{b\}} + \chi_T\right) = 0.$$



with T the c_j points in $\Gamma_1(a) \cap \Gamma_{j-1}(b)$.

Proof

Γ is regular near $2d$-gon of order (s, t) $(s > 1)$.
Suppose $d(a, b) = j$ and take $T = \Gamma_1(a) \cap \Gamma_{j-1}(b)$.
Consider $v \in \mathbb{R}^{\Omega}$:

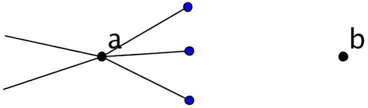
$$v = \alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T.$$

We need $v^T M v$ with

$$M = A_0 - \frac{A_1}{s} + \frac{A_2}{s^2} - \ldots + \frac{(-1)^d A_d}{s^d}.$$

Note that for any two subsets S_1, S_2:
$(\chi_{S_1})^T A_i \chi_{S_2}$ = number of pairs (x_1, x_2) in $(S_1 \times S_2)$ with $d(x_1, x_2) = i$.

Proof (continued)



Note: 2 distinct points in $T = \Gamma_1(a) \cap \Gamma_{j-1}(b)$ must be at distance 2. So with

$$v = \alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T, M = A_0 - \frac{A_1}{s} + \frac{A_2}{s^2} - \ldots + \frac{(-1)^d A_d}{s^d}.$$

we obtain that $v^T M v$ is $((\alpha, \beta, \gamma)^T F(\alpha, \beta, \gamma))/s^j$ with:

$$F = \begin{pmatrix} s^j & (-1)^j & -s^{j-1}c_j \\ (-1)^j & s^j & (-1)^{j-1}sc_j \\ -s^{j-1}c_j & (-1)^{j-1}sc_j & c_j s^{j-2}(s^2 + c_j - 1) \end{pmatrix}.$$

Proof (last part)

M is projection up to positive scalar, so $v^T M v \geq 0$ and thus $(\alpha, \beta, \gamma)^T F(\alpha, \beta, \gamma) \geq 0, \forall \alpha, \beta, \gamma \in \mathbb{R}$. Hence F is positive semidefinite:

$$\mathrm{Det}(F) = c_j s^{j-2}(s^2-1)((s^{2j}-1) - c_j(s^2-1)) \geq 0 \Longleftrightarrow c_j \leq \frac{s^{2j}-1}{s^2-1}.$$

$Mv = 0 \Longleftrightarrow v^T M v = 0$ if and only if $c_j = \frac{s^{2j}-1}{s^2-1}$ and (α, β, γ) is scalar multiple of:

$$\left(s\frac{s^{2j-2}-1}{s^2-1}, (-1)^j s^{j-1}, 1 \right).$$

□

Triple intersection numbers if SOME $c_j = \frac{s^{2j}-1}{s^2-1}$

- We know that if $d(a,b) = j$ and $(\alpha, \beta, \gamma) = \left(s\frac{s^{2j-2}-1}{s^2-1}, (-1)^j s^{j-1}, 1 \right)$:

$$M(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T) = 0.$$

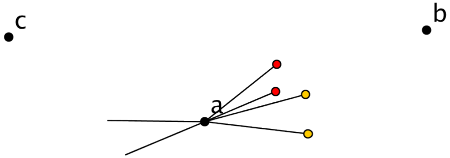
- For any vertex c we have:

$$(\chi_{\{c\}})^T M(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T) = 0.$$

(this is Delsarte's identity for outer distributions)

- If $d(a,c) = d$, the c_j points in $T = \Gamma_1(a) \cap \Gamma_{j-1}(b)$ are at distance $d-1$ or d from c. That identity allows us to compute how many of each!

Triple intersection numbers if SOME $c_j = \frac{s^{2j}-1}{s^2-1}$ (continued)



We assume $d(a,b) = j, d(a,c) = d, d(b,c) = k$.
Of the c_j points in $T = \Gamma_1(a) \cap \Gamma_{j-1}(b)$, precisely

$$\frac{s^{2j-1}-1}{s^2-1} + (-1)^{j+k+d} . \frac{s^{d-k+j-1}}{s+1}$$

are at distance $d-1$ from c, the rest is at distance d.

The case $d = j = k = 2$ from Introduction

If $c_2 = (s^4-1)/(s^2-1)$, we again find: $|\Gamma_1(a) \cap \Gamma_1(b) \cap \Gamma_1(c)| = s+1$.

Distance-regular last subconstituent in near $2d$-gon if ALL $c_j = \frac{s^{2j}-1}{s^2-1}$

Triple intersection numbers then yield:
subgraph $\Gamma_d(a)$ is also distance-regular for every point a.

The case $d \geq 3$

- The only regular near $2d$-gon Γ with such parameters is the unitary dual polar graph ${}^2A_{2d-1}(q)$.
- Last subconstituent $\Gamma_d(a)$ is isomorphic to *Hermitian forms graph*.

- A regular near $2d$-gon Γ of order (s, t) has $s+1$ points on each singular line.
- An *m-ovoid* is a subset of points S with m points of S on each singular line.
- If S is m-ovoid:
  - $a \in S$ has $(t+1)(m-1)$ neighbours in S,
  - $a \notin S$ has $(t+1)(m)$ neighbours in S.

  Hence S yields *equitable partition* of Γ into 2 parts.

Algebraic characterization of m-ovoids

- We know: if Γ is regular near $2d$-gon of order (s,t) and vertex set Ω:

  $$\mathbb{R}^{\Omega} = V_0 \perp \ldots \perp V_d,$$

  is decomposition into eigenspaces of Γ with:
  $V_0 = \langle \chi_\Omega \rangle$: eigenspace of largest eigenvalue $k = s(t+1)$,
  V_d: eigenspace of smallest eigenvalue $-(t+1)$.
- V_d is also kernel of incidence matrix from points to singular lines!
- S is m-ovoid $\iff \chi_S \in V_0 \perp V_d$.

Restricting m for m-ovoids S if some $c_j = (s^{2j} - 1)/(s^2 - 1), 2 \leq j \leq d$

Pick a and b in S with $d(a, b) = j$.
How many of the c_j in $T = \Gamma_1(a) \cap \Gamma_{j-1}(b)$ are also in S?

- $$v = \left(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T\right) \in (V_d)^{\perp},$$
  for certain $\alpha, \beta, \gamma \in \mathbb{R}$.
- If S is m-ovoid (i.e. every singular line meets S in m points):
  $$\chi_S \in (V_0 \perp V_d).$$
- So v and χ_S are *design-orthogonal* (Delsarte)
  Working out $\langle v, \chi_S\rangle$ yields $|\Gamma_1(a) \cap \Gamma_{j-1}(b) \cap S|$.
- Counting something in two ways then yields:
  $m = (s + 1)/2$ or $m = (s + 1)$ (trivial full set).

m-ovoids if $d = 2$ (so in generalized quadrangles of order (s, s^2))

- Note: *of order* $(s, t) = (s, s^2)$ means: $c_2 = (s^4 - 1)/(s^2 - 1)$.
- Restriction that $m = (s + 1)/2$ obtained by Segre (1965), Bruen-Hirschfeld (1978), Thas (1989).
- $((s + 1)/2)$-ovoid is called a *hemisystem* here.

Induced subgraph on $((s+1)/2)$-ovoid

- If Γ is regular near $2d$-gon of order (s,t) with $s>1$ and ALL $c_j = \frac{s^{2j}-1}{s^2-1}$, then $((s+1)/2)$-ovoid is distance-regular subgraph with:

$$b'_j = \frac{s^{2d}-s^{2j}}{2(s+1)};\ c'_j = \frac{(s^j-(-1)^j)(s^{j-1}-(-1)^j)}{2(s+1)}.$$

- If for a general distance-regular graph Γ:

$$b_j = \frac{b^d-b^j}{b-1}\left(\beta-\alpha\frac{b^j-1}{b-1}\right);\ c_j = \frac{b^j-1}{b-1}\left(1+\alpha\frac{b^{j-1}-1}{b-1}\right).$$

  we say Γ has *classical parameters* (d,b,α,β).

- So here we find classical parameters $(d,-s,-(s+1)/2,-((-s)^d+1)/2)$.

The case $d = 2$: generalized quadrangles of order (s, s^2)

- $((s+1)/2)$-ovoid (or hemisystem) yields strongly regular graph: $srg(v, k, \lambda, \mu) =$ $srg((s+1)(s^3+1)/2, (s-1)(s^2+1)/2, (s-3)/2, (s-1)^2/2)$.
- Thas (for ${}^2A_3(q)$) and Cameron-Delsarte-Goethals (1979)

The case $d \geq 3$

- Only unitary dual polar graph ${}^2A_{2d-1}(q)$ with odd $s = q$ is suitable.
- This would yield subgraph with classical parameters: $(d, -q, -(q+1)/2, -((-q)^d+1)/2)$.

For any $d \geq 2$, subgraph is triangle-free for $s = 3$.

Meaning of those parameters $(d, -q, -(q+1)/2, -((-q)^d+1)/2)$

Weng (1998):
If Γ has classical parameters (d, b, α, β) with $b < -1$
and $d \geq 4, c_2 > 1$ and has triangles,
then $-b$ is odd prime power q and either:

- Γ is unitary dual polar graph ${}^2A_{2d-1}(q)$,
- Γ is Hermitian forms graph or thus last subconstituent of ${}^2A_{2d-1}(q)$,
- $(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d+1)/2)$.

No examples of last type for $d = 3$ or $d \geq 4$ seem known,
but $((q+1)/2)$-ovoid of ${}^2A_{2d-1}(q)$ would yield it.

Can we actually find these $((q+1)/2)$-ovoids in ${}^2A_{2d-1}(q)$?

Remember about dual polar graph:

- isotropic d- and $(d-1)$-spaces
  are points and singular lines, respectively.
  (Incidence is just symmetrized inclusion.)

Hence S *is m-ovoid in dual polar graph* means:
every isotropic $(d-1)$-space is in m of the d-spaces in S.
This fits into Delsarte's theory of t-designs in *regular semilattices*!
See Stanton (1986) and Munemasa (1986) for overview.

t-designs S of k-sets in set of size n

- S is t-design here means : every t-set in exactly λ elements of S.

t-designs S of k-spaces in vector space $V(n, q)$

- S is t-design here means: every t-space in exactly λ elements of S.
- 1-designs are not so hard (use for instance Singer cycles).
- first non-trivial 2-designs by Thomas (1987): in $V(6m \pm 1, 2)$, more constructions by Suzuki (1990,1992) and Itoh (1998).
- one sporadic 3-design by Braun-Kerber-Laue (2003): in $V(8, 2)$.

t-designs S of isotropic d-spaces in polar spaces of rank d

- S is t-design here means:
  every isotropic t-space in exactly λ elements of S,
- A t-design is automatically also 1-design, 2-design,...,$(t-1)$-design.
- 1-designs are not so rare (for instance the *spreads*),
- No non-trivial t-designs with $t \geq 2$ seem to be known.
- But we want m-ovoids, so $(d-1)$-designs!

Non-trivial $(d-1)$-designs S of isotropic d-spaces in ${}^2A_{2d-1}(q)$ for odd q?

- S is $(d-1)$-design here means: every isotropic $(d-1)$-space in exactly $(q+1)/2$ of the d-spaces in S,
- For $d=2$ (so hemisystem), it exists for $q=3$ (Segre (1965)), yielding (triangle-free) strongly regular *Gewirtz graph* $srg(56, 10, 0, 2)$.
- For $d=2$ (so hemisystem), they exist for ALL odd q (Cossidente-Penttila (2005)), yielding strongly regular $srg((q+1)(q^3+1)/2, (q-1)(q^2+1)/2, (q-3)/2, (q-1)^2/2)$.
- Existence in ${}^2A_{2d-1}(q)$ also implies existence in ${}^2A_{2(d-1)-1}(q)$, so $d=3$ is next place to look.
- 1-designs of right size (half) exist for $d=3$ and all odd q (Cossidente-Penttila (2009)).
- Conjecture by Lu-Pan-Weng (2008): *triangle-free Gewirtz graph does NOT grow*, so no such $(d-1)$-designs for $d \geq 3$ and $q=3$?

Thank you for your attention!

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