

An algebraic approach to subsets in association schemes from finite buildings

Frédéric Vanhove
Department of Pure Mathematics and Computer Algebra
Ghent University
fvanhove@cage.ugent.be
<http://cage.ugent.be/fvanhove>

ALCOMA10
April 12, 2010

Outline

- Algebraic background: which techniques do we need? (Philippe Delsarte)
- Buildings: what are these structures? (Jacques Tits)
- Buildings of type F_4 : some results!

Definition of association scheme

A d -class association scheme is a pair $(\Omega, \{R_0, \dots, R_d\})$ with Ω a set and R_0, \dots, R_d symmetric relations on Ω s.t.:

Definition of association scheme

A d -class association scheme is a pair $(\Omega, \{R_0, \dots, R_d\})$ with Ω a set and R_0, \dots, R_d symmetric relations on Ω s.t.:

- (i) $R_0 =$ identity relation,

Definition of association scheme

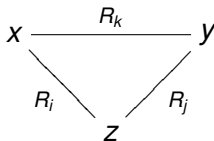
A d -class association scheme is a pair $(\Omega, \{R_0, \dots, R_d\})$ with Ω a set and R_0, \dots, R_d symmetric relations on Ω s.t.:

- (i) $R_0 =$ identity relation,
- (ii) $\{R_0, \dots, R_d\}$ is a partition of $\Omega \times \Omega$,

Definition of association scheme

A d -class association scheme is a pair $(\Omega, \{R_0, \dots, R_d\})$ with Ω a set and R_0, \dots, R_d symmetric relations on Ω s.t.:

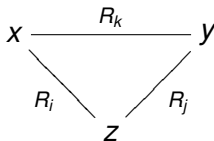
- (i) $R_0 =$ identity relation,
- (ii) $\{R_0, \dots, R_d\}$ is a partition of $\Omega \times \Omega$,
- (iii) there are *intersection numbers* p_{ij}^k :
if $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is p_{ij}^k .



Definition of association scheme

A d -class association scheme is a pair $(\Omega, \{R_0, \dots, R_d\})$ with Ω a set and R_0, \dots, R_d symmetric relations on Ω s.t.:

- (i) $R_0 =$ identity relation,
- (ii) $\{R_0, \dots, R_d\}$ is a partition of $\Omega \times \Omega$,
- (iii) there are *intersection numbers* p_{ij}^k :
if $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is p_{ij}^k .



Every relation R_i is thus regular, with valency $k_i = p_{ii}^0$.

Example: the Paley scheme $P_5 = (\Omega, \{R_0, R_1, R_2\})$

- $\Omega = \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$.

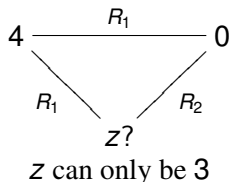
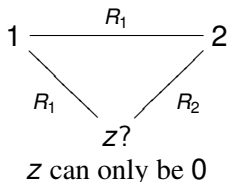
Example: the Paley scheme $P_5 = (\Omega, \{R_0, R_1, R_2\})$

- $\Omega = \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$.
- We define 3 relations R_0, R_1, R_2 :
 - R_0 is the identity relation: e.g. $(2, 2) \in R_0$,
 - $(a, b) \in R_1$ if $a - b$ is 1 or 4 (and hence square): e.g. $(2, 3) \in R_1$,
 - $(a, b) \in R_2$ if $a - b$ is 2 or 3 (and hence non-square): e.g. $(1, 4) \in R_2$.

Example: the Paley scheme $P_5 = (\Omega, \{R_0, R_1, R_2\})$

- $\Omega = \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$.
- We define 3 relations R_0, R_1, R_2 :
 - R_0 is the identity relation: e.g. $(2, 2) \in R_0$,
 - $(a, b) \in R_1$ if $a - b$ is 1 or 4 (and hence square): e.g. $(2, 3) \in R_1$,
 - $(a, b) \in R_2$ if $a - b$ is 2 or 3 (and hence non-square): e.g. $(1, 4) \in R_2$.

Intersection number p_{12}^1 is 1:



Definition of matrices A_i

Consider an association scheme $(\Omega, \{R_0, \dots, R_d\})$ and order the elements of Ω : $\omega_1, \dots, \omega_{|\Omega|}$.

Definition of matrices A_i

Consider an association scheme $(\Omega, \{R_0, \dots, R_d\})$ and order the elements of Ω : $\omega_1, \dots, \omega_{|\Omega|}$.

For each relation R_i , define the $(|\Omega| \times |\Omega|)$ -matrix A_i over \mathbb{R} :

$$\begin{cases} (A_i)_{rs} = 1 & \text{if } (\omega_r, \omega_s) \in R_i, \\ (A_i)_{rs} = 0 & \text{if } (\omega_r, \omega_s) \notin R_i. \end{cases}$$

Definition of matrices A_i

Consider an association scheme $(\Omega, \{R_0, \dots, R_d\})$ and order the elements of Ω : $\omega_1, \dots, \omega_{|\Omega|}$.

For each relation R_i , define the $(|\Omega| \times |\Omega|)$ -matrix A_i over \mathbb{R} :

$$\begin{cases} (A_i)_{rs} = 1 & \text{if } (\omega_r, \omega_s) \in R_i, \\ (A_i)_{rs} = 0 & \text{if } (\omega_r, \omega_s) \notin R_i. \end{cases}$$

Properties

- A_0 is the identity matrix.
- $A_0 + \dots + A_d$ is all-one matrix.
- A_i is symmetric.
- $A_i A_j = \sum_k p_{ij}^k A_k$.

Example: Paley scheme $P_5 = (\{0, 1, 2, 3, 4\}, \{R_0, R_1, R_2\})$

Example: Paley scheme $P_5 = (\{0, 1, 2, 3, 4\}, \{R_0, R_1, R_2\})$

$$A_0 = \begin{array}{ccccc} & 0 & 1 & 2 & 3 & 4 \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & 0 & 1 & 2 & 3 & 4 \end{array}$$

$$A_1 = \begin{array}{ccccc} & 0 & 1 & 2 & 3 & 4 \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} & 0 & 1 & 2 & 3 & 4 \end{array}, A_2 = \begin{array}{ccccc} & 0 & 1 & 2 & 3 & 4 \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} & 0 & 1 & 2 & 3 & 4 \end{array}$$

Note that A_0, A_1 and A_2 are symmetric and add up to the all-one matrix!

Decomposition into strata for scheme $(\Omega, \{R_0, \dots, R_d\})$

$\mathbb{R}\Omega$ is the vector space with basis indexed by elements of Ω .

Decomposition into strata for scheme $(\Omega, \{R_0, \dots, R_d\})$

$\mathbb{R}\Omega$ is the vector space with basis indexed by elements of Ω .

$\mathbb{R}\Omega$ uniquely decomposes as:

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d,$$

with every $v \in V_i$ an eigenvector of every relation R_j : $A_j v = \lambda_{ij} v$.
These V_i are the strata, and by convention $V_0 = \langle \chi_\Omega \rangle$.

Decomposition into strata for scheme $(\Omega, \{R_0, \dots, R_d\})$

$\mathbb{R}\Omega$ is the vector space with basis indexed by elements of Ω .

$\mathbb{R}\Omega$ uniquely decomposes as:

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d,$$

with every $v \in V_i$ an eigenvector of every relation R_j : $A_j v = \lambda_{ij} v$.
These V_i are the strata, and by convention $V_0 = \langle \chi_\Omega \rangle$.

Matrix of eigenvalues P

$$P = \begin{array}{c|cccc} & R_0 & R_1 & \dots & R_d \\ \hline V_0 & 1 & \lambda_{01} & \dots & \lambda_{0d} \\ V_1 & 1 & \ddots & \ddots & \lambda_{1d} \\ \vdots & 1 & \ddots & \ddots & \vdots \\ V_d & 1 & \lambda_{d1} & \dots & \lambda_{dd} \end{array}$$

Counting algebraically

The characteristic vector χ_S of $S \subseteq \Omega$ has :

$(\chi_S)_i = 1$ if $\omega_i \in S$ and $(\chi_S)_i = 0$ if $\omega_i \notin S$. So $\chi_S = (0, 1, 0, 1, 1, \dots)$.

Counting algebraically

The characteristic vector χ_S of $S \subseteq \Omega$ has :

$(\chi_S)_i = 1$ if $\omega_i \in S$ and $(\chi_S)_i = 0$ if $\omega_i \notin S$. So $\chi_S = (0, 1, 0, 1, 1, \dots)$.

For any $S, S' \subseteq \Omega$: $\langle \chi_S, \chi_{S'} \rangle = (1, 1, \dots, 1)(0, 1, \dots, 0) = |S \cap S'|$.

Counting algebraically

The characteristic vector χ_S of $S \subseteq \Omega$ has :

$(\chi_S)_i = 1$ if $\omega_i \in S$ and $(\chi_S)_i = 0$ if $\omega_i \notin S$. So $\chi_S = (0, 1, 0, 1, 1, \dots)$.

For any $S, S' \subseteq \Omega$: $\langle \chi_S, \chi_{S'} \rangle = (1, 1, \dots, 1)(0, 1, \dots, 0) = |S \cap S'|$.

Design-orthogonality of subsets S and S'

If $\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d$ and:

$$\chi_S = \frac{|S|}{|\Omega|} \chi_\Omega + \cancel{v_2} + \dots + v_d, v_i \in V_i$$

$$\chi_{S'} = \frac{|S'|}{|\Omega|} \chi_\Omega + \cancel{v_1} + v'_2 + \dots + \cancel{v_d}, v'_i \in V_i$$

with $\forall i \in \{1, \dots, d\}$: $v_i = 0$ or $v'_i = 0$ then

$$|S \cap S'| = \langle \chi_S, \chi_{S'} \rangle = \frac{|S||S'|}{|\Omega||\Omega|} \langle \chi_\Omega, \chi_\Omega \rangle = \frac{|S||S'|}{|\Omega|}$$

Which components v_i of χ_S are zero?

The inner distribution \mathbf{a} of $S \subseteq \Omega$: $\mathbf{a}_i = \frac{1}{|S|} |(S \times S) \cap R_i|$.

Which components v_i of χ_S are zero?

The inner distribution \mathbf{a} of $S \subseteq \Omega$: $\mathbf{a}_i = \frac{1}{|S|} |(S \times S) \cap R_i|$.

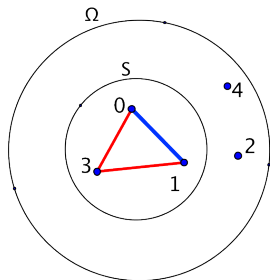
So \mathbf{a}_i is average valency of R_i in S , and $\mathbf{a}_0 = 1$, $\mathbf{a}_0 + \dots + \mathbf{a}_d = |S|$.

Which components v_i of χ_S are zero?

The inner distribution \mathbf{a} of $S \subseteq \Omega$: $\mathbf{a}_i = \frac{1}{|S|} |(S \times S) \cap R_i|$.

So \mathbf{a}_i is average valency of R_i in S , and $\mathbf{a}_0 = 1$, $\mathbf{a}_0 + \dots + \mathbf{a}_d = |S|$.

Example: Paley scheme $(\Omega, \{R_0, R_1, R_2\})$, and $S = \{0, 1, 3\}$.



$$\mathbf{a} = \frac{1}{3} \left((1, 1, 1)_0 + (1, 1, 1)_1 + (1, 0, 2)_3 \right) = \left(1, \frac{2}{3}, \frac{4}{3} \right).$$

Which components v_i of χ_S are zero?

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d$$

$$\chi_S = \frac{|S|}{|\Omega|} \chi_\Omega + v_1 + \dots + v_d, \text{ with } v_i \in V_i.$$

Which components v_i of χ_S are zero?

$$\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d$$

$$\chi_S = \frac{|S|}{|\Omega|} \chi_\Omega + v_1 + \dots + v_d, \text{ with } v_i \in V_i.$$

For any (non-empty) subset $S \subseteq \Omega$, inner distribution \mathbf{a} satisfies:

$$(\mathbf{a}P^{-1})_i \geq 0,$$

with equality if and only if $v_i = 0$ (or hence $\chi_S \in (V_i)^\perp$).

The Johnson scheme $J(n, k)$ in a set X of size n

- Ω : the $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 1}$ k -subsets in X ,
- $(\omega_1, \omega_2) \in R_i \iff |\omega_1 \cap \omega_2| = k - i$.

The Johnson scheme $J(n, k)$ in a set X of size n

- Ω : the $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 1}$ k -subsets in X ,
- $(\omega_1, \omega_2) \in R_i \iff |\omega_1 \cap \omega_2| = k - i$.

The Grassmann scheme $J_q(n, k)$ in $V(n, q)$

- Ω : the $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)}$ k -subspaces in $V(n, q)$,
- $(\omega_1, \omega_2) \in R_i \iff \dim(\omega_1 \cap \omega_2) = k - i$.

Delsarte's characterization in $J(n, k)$ and $J_q(n, k)$

- In both schemes: $\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d$ with $d = \min(k, n - k)$ (with “natural ordering”).

Delsarte's characterization in $J(n, k)$ and $J_q(n, k)$

- In both schemes: $\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d$ with $d = \min(k, n - k)$ (with “natural ordering”).
- In $J(n, k)$: $S \subseteq \Omega$ is $t - (n, k, \lambda)$ -design (i.e. every t -subset in X is in λ elements of S)
 $\iff \chi_S \in V_0 \perp \cancel{V_1} \perp \dots \perp \cancel{V_t} \perp V_{t+1} \perp \dots \perp V_d.$

Delsarte's characterization in $J(n, k)$ and $J_q(n, k)$

- In both schemes: $\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d$ with $d = \min(k, n - k)$ (with “natural ordering”).
- In $J(n, k)$: $S \subseteq \Omega$ is $t - (n, k, \lambda)$ -design
(i.e. every t -subset in X is in λ elements of S)
 $\iff \chi_S \in V_0 \perp \cancel{V_1} \perp \dots \perp \cancel{V_t} \perp V_{t+1} \perp \dots \perp V_d$.
- In $J_q(n, k)$: $S \subseteq \Omega$ is $t - (n, k, \lambda; q)$ -design
(i.e. every t -space in $V(n, q)$ is in λ elements of S)
 $\iff \chi_S \in V_0 \perp \cancel{V_1} \perp \dots \perp \cancel{V_t} \perp V_{t+1} \perp \dots \perp V_d$.

Delsarte's characterization in $J(n, k)$ and $J_q(n, k)$

- In both schemes: $\mathbb{R}\Omega = V_0 \perp V_1 \perp \dots \perp V_d$ with $d = \min(k, n - k)$ (with “natural ordering”).
- In $J(n, k)$: $S \subseteq \Omega$ is $t - (n, k, \lambda)$ -design
(i.e. every t -subset in X is in λ elements of S)
 $\iff \chi_S \in V_0 \perp \cancel{V_1} \perp \dots \perp \cancel{V_t} \perp V_{t+1} \perp \dots \perp V_d$.
- In $J_q(n, k)$: $S \subseteq \Omega$ is $t - (n, k, \lambda; q)$ -design
(i.e. every t -space in $V(n, q)$ is in λ elements of S)
 $\iff \chi_S \in V_0 \perp \cancel{V_1} \perp \dots \perp \cancel{V_t} \perp V_{t+1} \perp \dots \perp V_d$.

Where else do we have these characterizations in schemes and their q -analogues?

Buildings of rank n are certain incidence structures with n types (Tits).

Buildings of rank n are certain incidence structures with n types (Tits).
Each finite building has a Coxeter-Dynkin diagram, for instance:



- Each node corresponds with a type (points, lines, planes, ...).

Buildings of rank n are certain incidence structures with n types (Tits).
Each finite building has a Coxeter-Dynkin diagram, for instance:



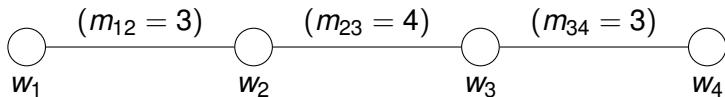
- Each node corresponds with a type (points, lines, planes, ...).
- If $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ of types $1, \dots, i-1, i+1, \dots, n$ are incident, they are incident with $q_i + 1$ common objects of type i .

Buildings of rank n are certain incidence structures with n types (Tits).
Each finite building has a Coxeter-Dynkin diagram, for instance:



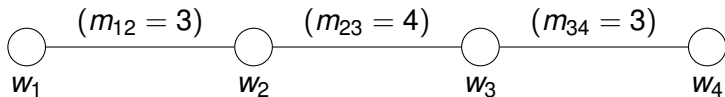
- Each node corresponds with a type (points, lines, planes, ...).
- If $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ of types $1, \dots, i-1, i+1, \dots, n$ are incident, they are incident with $q_i + 1$ common objects of type i .
- The thin building with $(q_1, \dots, q_n) = (1, \dots, 1)$ is constructed from "Coxeter group" itself, thick building is more complicated but similar.

Coxeter group W and its thin building



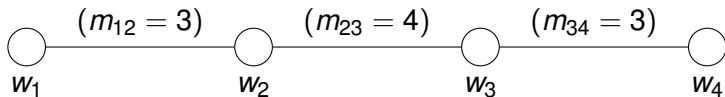
- W is generated by involutions w_i , with $(w_i w_j)^{m_{ij}} = e$
(we draw no line if $m_{ij} = 2 \iff w_i w_j = w_j w_i$),

Coxeter group W and its thin building



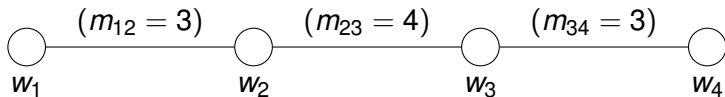
- W is generated by involutions w_i , with $(w_i w_j)^{m_{ij}} = e$
(we draw no line if $m_{ij} = 2 \iff w_i w_j = w_j w_i$),
- maximal parabolic subgroup $P_i := \langle \{w_1, \dots, w_n\} \setminus \{w_i\} \rangle$,

Coxeter group W and its thin building



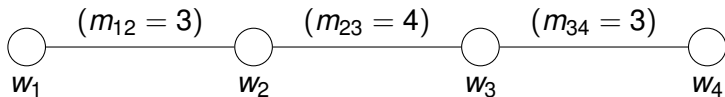
- W is generated by involutions w_i , with $(w_i w_j)^{m_{ij}} = e$
(we draw no line if $m_{ij} = 2 \iff w_i w_j = w_j w_i$),
- maximal parabolic subgroup $P_i := \langle \{w_1, \dots, w_n\} \setminus \{w_i\} \rangle$,
- objects of type i : cosets wP_i ,

Coxeter group W and its thin building



- W is generated by involutions w_i , with $(w_i w_j)^{m_{ij}} = e$
(we draw no line if $m_{ij} = 2 \iff w_i w_j = w_j w_i$),
- maximal parabolic subgroup $P_i := \langle \{w_1, \dots, w_n\} \setminus \{w_i\} \rangle$,
- objects of type i : cosets wP_i ,
- aP_i and bP_j are incident iff $aP_i \cap bP_j \neq \emptyset$,

Coxeter group W and its thin building



- W is generated by involutions w_i , with $(w_i w_j)^{m_{ij}} = e$
(we draw no line if $m_{ij} = 2 \iff w_i w_j = w_j w_i$),
- maximal parabolic subgroup $P_i := \langle \{w_1, \dots, w_n\} \setminus \{w_i\} \rangle$,
- objects of type i : cosets wP_i ,
- aP_i and bP_j are incident iff $aP_i \cap bP_j \neq \emptyset$,
- (aP_i, bP_j) and $(a'P_i, b'P_j)$ in same relation iff
 $P_i a^{-1} b P_j = P_i a'^{-1} b' P_j$
(if $i = j$, this is an association scheme in SOME cases!)



$A_n \cong$ symmetric group of size $(n + 1)!$ on $n + 1$ elements.



$A_n \cong$ symmetric group of size $(n + 1)!$ on $n + 1$ elements.

Thin buildings of type A_n , with $q_1 = \dots = q_n = 1 \rightarrow$ Johnson scheme!

- objects of type i : the $\binom{n+1}{i}$ subsets of size i ,
- 2 objects of type i are k -related iff size intersection is $i - k$.



$A_n \cong$ symmetric group of size $(n + 1)!$ on $n + 1$ elements.

Thin buildings of type A_n , with $q_1 = \dots = q_n = 1 \rightarrow$ Johnson scheme!

- objects of type i : the $\binom{n+1}{i}$ subsets of size i ,
- 2 objects of type i are k -related iff size intersection is $i - k$.

Thick buildings of type A_n , with $q_1 = \dots = q_n = q \rightarrow$ Grassmann scheme!

- objects of type i : the $\left[\begin{smallmatrix} n+1 \\ i \end{smallmatrix} \right]_q$ i -spaces of $V(n + 1, q)$,
- two objects of type i are k -related iff dim intersection is $i - k$.



$B_n \cong$ hyperoctahedral group of size $2^n n!$



$B_n \cong$ hyperoctahedral group of size $2^n n!$

Thin building, with $q_1 = \dots = q_n = 1$ \rightarrow coding theory!

- type n : 2^n binary codewords of length n ,
- 2 objects of type n are k -related iff they are equal in $n - k$ positions.



$B_n \cong$ hyperoctahedral group of size $2^n n!$

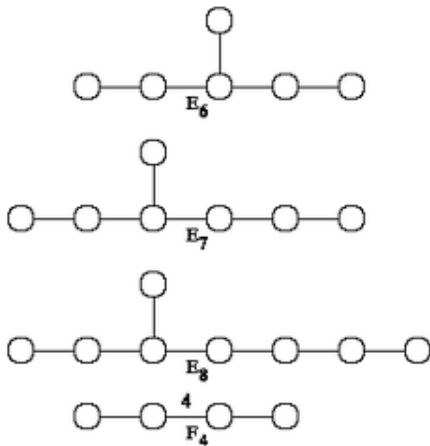
Thin building, with $q_1 = \dots = q_n = 1$ \rightarrow coding theory!

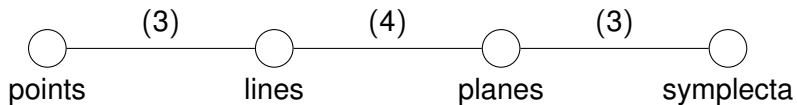
- type n : 2^n binary codewords of length n ,
- 2 objects of type n are k -related iff they are equal in $n - k$ positions.

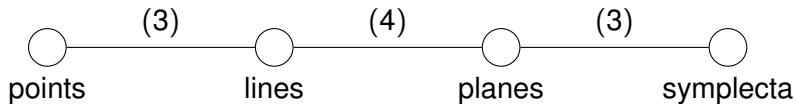
Thick building, with $q_1 = \dots = q_{n-1} = q$, $q_n = q^e$ \rightarrow polar spaces!

- type n : $(q^e + 1) \dots (q^{n-1+e} + 1)$ maximals of rank n polar space,
- 2 maximals are k -related iff dim intersection is $n - k$.

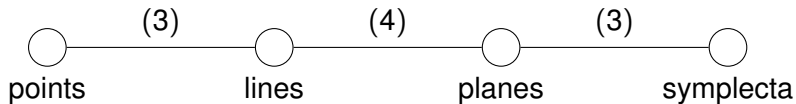
What about other finite Coxeter groups?





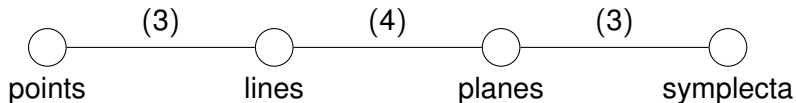


Thin building of type F_4



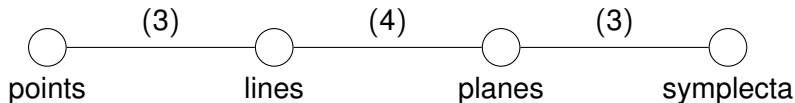
Thin building of type F_4

- $|F_4| = 1152 = 2^7 \cdot 3^2$,
- 24 points, 96 lines and planes, 24 symplecta.



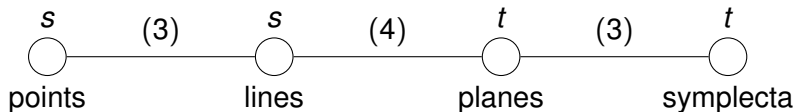
Thin building of type F_4

- $|F_4| = 1152 = 2^7 \cdot 3^2$,
- 24 points, 96 lines and planes, 24 symplecta.
- Points: all 24 vectors in \mathbb{R}^4 of form $(\pm 1, \pm 1, 0, 0)$ up to permutation.



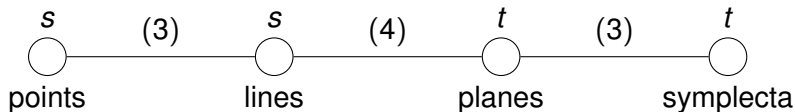
Thin building of type F_4

- $|F_4| = 1152 = 2^7 \cdot 3^2$,
- 24 points, 96 lines and planes, 24 symplecta.
- Points: all 24 vectors in \mathbb{R}^4 of form $(\pm 1, \pm 1, 0, 0)$ up to permutation.
- Relations between points: R_0, R_1, R_2, R_3, R_4 correspond with inner products 2, 1, 0, $-1, -2$ (24-cell).



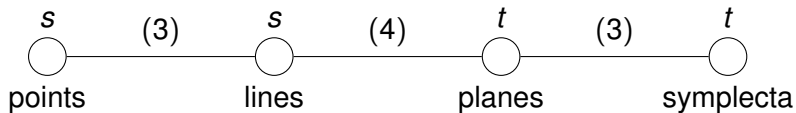
Thick building of type F_4

- $(s, t) = (q, 1), (1, q), (q, q), (q, q^2)$ or (q^2, q) ,
- number of points: $|\Omega| = (s^2 + s + 1)(s^2t + 1)(s^2t^2 + 1)(s^3t^3 + 1)$,

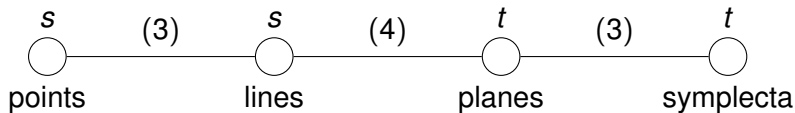


Thick building of type F_4

- $(s, t) = (q, 1), (1, q), (q, q), (q, q^2)$ or (q^2, q) ,
- number of points: $|\Omega| = (s^2 + s + 1)(s^2t + 1)(s^2t^2 + 1)(s^3t^3 + 1)$,
- association scheme on points:
 - R_0 : identity,
 - R_1 : on unique common line (collinear),
 - R_2 : in unique common symplecton (cohyperlinear),
 - R_3 : unique common neighbour wrt collinearity (almost opposite),
 - R_4 : no common neighbour (opposite).

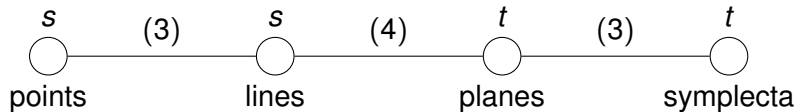


Matrix of eigenvalues P of association scheme on points (Gomi):
 $\mathbb{R}\Omega$ decomposes into strata: $V_0 \perp V_1 \perp V_2 \perp V_3 \perp V_4$.



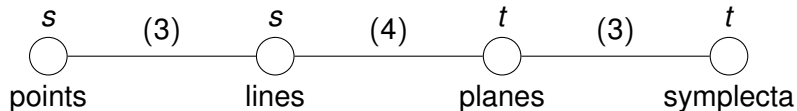
Matrix of eigenvalues P of association scheme on points (Gomi):
 $\mathbb{R}\Omega$ decomposes into strata: $V_0 \perp V_1 \perp V_2 \perp V_3 \perp V_4$.

	R_0	R_1	R_2	R_3	R_4
V_0	1	$s(s+1)(st+1)(st^2+1)$	$s^4t(t^2+t+1)(st^2+1)$	$s^5t^3(s+1)(st+1)(st^2+1)$	s^9t^6
V_1	1	$(s+1)(st+1)(st+s-1)$	$s^2(t^2+t+1)(s^2t-1)$	$s^3t(s+1)(st+1)(st-t-1)$	$-s^6t^3$
V_2	1	$(-1+s^2)(st+1)$	$(s^3t-s^2t^2-s^2t-st-s+t)s$	$-(s-1)(s+1)(st+1)s^2t$	s^5t^2
V_3	1	$-1-st-st^2+s^2$	$s(t-s)(t^2+t+1)$	$st(s^2t^2-t^2+st+s)$	$-s^3t^3$
V_4	1	$-(st+1)(st^2+1)$	$st(t^2+t+1)(st^2+1)$	$-(st+1)(st^2+1)st^3$	t^6s^3



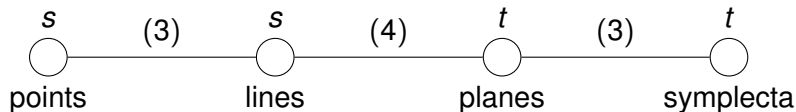
The set of points S inside an object

- points in plane interact as in projective plane (so all collinear):
inner distribution of S : $\mathbf{a} = (1, s^2 + s, 0, 0, 0)$,



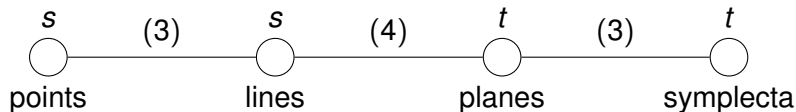
The set of points S inside an object

- points in plane interact as in projective plane (so all collinear):
inner distribution of S : $\mathbf{a} = (1, s^2 + s, 0, 0, 0)$,
 $\mathbf{a}P^{-1} = (?, ?, ?, ?, 0) \implies \chi_S \in V_0 \perp V_1 \perp V_2 \perp V_3 \perp \cancel{V_4}$.



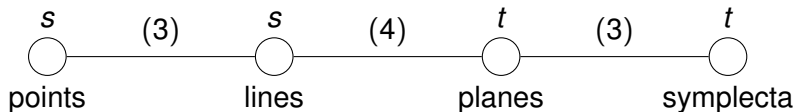
The set of points S inside an object

- points in plane interact as in projective plane (so all collinear):
inner distribution of S : $\mathbf{a} = (1, s^2 + s, 0, 0, 0)$,
 $\mathbf{a}P^{-1} = (?, ?, ?, ?, 0) \implies \chi_S \in V_0 \perp V_1 \perp V_2 \perp V_3 \perp \cancel{V_4}$.
- points in symplecton interact as in rank 3 polar space (collinear or cohyperlinear):
inner distribution of S : $\mathbf{a} = (1, s(s+1)(st+1), s^4t, 0, 0)$.



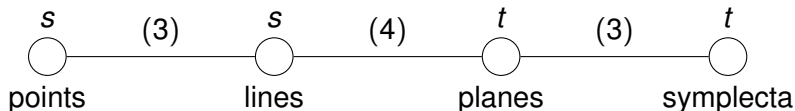
The set of points S inside an object

- points in plane interact as in projective plane (so all collinear):
 inner distribution of S : $\mathbf{a} = (1, s^2 + s, 0, 0, 0)$,
 $\mathbf{a}P^{-1} = (?, ?, ?, ?, 0) \implies \chi_S \in V_0 \perp V_1 \perp V_2 \perp V_3 \perp \cancel{V_4}$.
- points in symplecton interact as in rank 3 polar space
 (collinear or cohyperlinear):
 inner distribution of S : $\mathbf{a} = (1, s(s+1)(st+1), s^4t, 0, 0)$.
 $\mathbf{a}P^{-1} = (?, ?, ?, 0, 0) \implies \chi_S \in V_0 \perp V_1 \perp V_2 \perp \cancel{V_3} \perp \cancel{V_4}$.



Notion of design T in F_4 -building

- Set of points S in any plane : $\chi_S \in V_0 \perp V_1 \perp V_2 \perp V_3 \perp \cancel{V_4}$,
So if $\chi_T \in V_0 \perp \cancel{V_1} \perp \cancel{V_2} \perp \cancel{V_3} \perp V_4$
 $\implies |S \cap T| = \frac{|S||T|}{|\Omega|} = |S| / ((s^2t + 1)(s^2t^2 + 1)(s^3t^3 + 1)).$



Notion of design T in F_4 -building

- Set of points S in any plane : $\chi_S \in V_0 \perp V_1 \perp V_2 \perp V_3 \perp \cancel{V_4}$,
 So if $\chi_T \in V_0 \perp \cancel{V_1} \perp \cancel{V_2} \perp \cancel{V_3} \perp V_4$
 $\implies |S \cap T| = \frac{|S||T|}{|\Omega|} = |S| / ((s^2t + 1)(s^2t^2 + 1)(s^3t^3 + 1))$.
- Set of points S in any symplecton: $\chi_S \in V_0 \perp V_1 \perp V_2 \perp \cancel{V_3} \perp \cancel{V_4}$,
 So if $\chi_T \in V_0 \perp \cancel{V_1} \perp \cancel{V_2} \perp V_3 \perp V_4$
 $\implies |S \cap T| = \frac{|S||T|}{|\Omega|} = |S| / ((s^2t^2 + 1)(s^3t^3 + 1))$.

Cliques of the oppositeness-relation R_4

If S is a set of mutually opposite points (hence not collinear with a common point):

$$\mathbf{a} = (1, 0, 0, 0, |S| - 1).$$

Cliques of the oppositeness-relation R_4

If S is a set of mutually opposite points (hence not collinear with a common point):

$$\mathbf{a} = (1, 0, 0, 0, |S| - 1).$$

$\mathbb{R}\Omega = V_0 \perp V_1 \perp V_2 \perp V_3 \perp V_4$, matrix of eigenvalues P :

	R_0	R_1	R_2	R_3	R_4
V_0	1	$s(s+1)(st+1)(st^2+1)$	$s^4 t(t^2+t+1)(st^2+1)$	$s^5 t^3(s+1)(st+1)(st^2+1)$	$s^9 t^6$
V_1	1	$(s+1)(st+1)(st+s-1)$	$s^2(t^2+t+1)(s^2t-1)$	$s^3 t(s+1)(st+1)(st-t-1)$	$-s^6 t^3$
V_2	1	$(-1+s^2)(st+1)$	$(s^3 t - s^2 t^2 - s^2 t - st - s + t) s$	$-(s-1)(s+1)(st+1)s^2 t$	$s^5 t^2$
V_3	1	$-1 - st - st^2 + s^2$	$s(t-s)(t^2+t+1)$	$st(s^2 t^2 - t^2 + st + s)$	$-s^3 t^3$
V_4	1	$-(st+1)(st^2+1)$	$st(t^2+t+1)(st^2+1)$	$-(st+1)(st^2+1)st^3$	$t^6 s^3$

- $(\mathbf{a}P^{-1})_i \geq 0$ with equality iff $\chi_S \in (V_i)^\perp$.

Cliques of the oppositeness-relation R_4

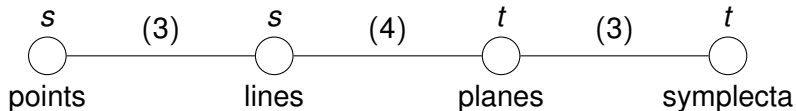
If S is a set of mutually opposite points (hence not collinear with a common point):

$$\mathbf{a} = (1, 0, 0, 0, |S| - 1).$$

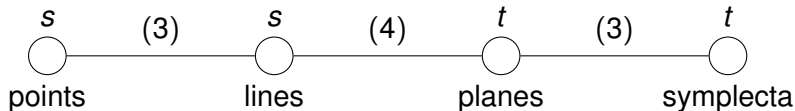
$\mathbb{R}\Omega = V_0 \perp V_1 \perp V_2 \perp V_3 \perp V_4$, matrix of eigenvalues P :

	R_0	R_1	R_2	R_3	R_4
V_0	1	$s(s+1)(st+1)(st^2+1)$	$s^4 t(t^2+t+1)(st^2+1)$	$s^5 t^3(s+1)(st+1)(st^2+1)$	$s^9 t^6$
V_1	1	$(s+1)(st+1)(st+s-1)$	$s^2(t^2+t+1)(s^2 t-1)$	$s^3 t(s+1)(st+1)(st-t-1)$	$-s^6 t^3$
V_2	1	$(-1+s^2)(st+1)$	$(s^3 t - s^2 t^2 - s^2 t - st - s + t) s$	$-(s-1)(s+1)(st+1)s^2 t$	$s^5 t^2$
V_3	1	$-1 - st - st^2 + s^2$	$s(t-s)(t^2+t+1)$	$st(s^2 t^2 - t^2 + st + s)$	$-s^3 t^3$
V_4	1	$-(st+1)(st^2+1)$	$st(t^2+t+1)(st^2+1)$	$-(st+1)(st^2+1)st^3$	$t^6 s^3$

- $(\mathbf{a}P^{-1})_i \geq 0$ with equality iff $\chi_S \in (V_i)^\perp$.
- $i = 1$ yields best bound: $|S| \leq s^3 t^3 + 1$, with equality iff $\chi_S \in V_0 \perp \cancel{V_1} \perp V_2 \perp V_3 \perp V_4$.

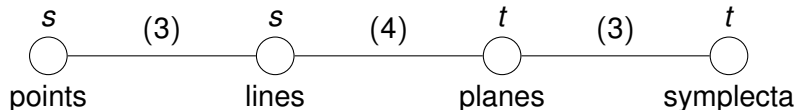
Embedding of one F_4 -building in another

Embed points T of F_4 -building with parameters (s', t') in this building:
(inner distribution \mathbf{a} of T consists just of valencies in smallest building)

Embedding of one F_4 -building in another

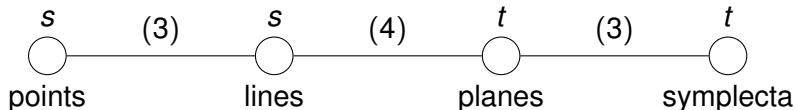
Embed points T of F_4 -building with parameters (s', t') in this building:
(inner distribution \mathbf{a} of T consists just of valencies in smallest building)

- if $(s', t') = (q, 1)$, $(s, t) = (q, q)$, then
 $\chi_T \in V_0 \perp V_1 \perp V_2 \perp \cancel{V_3} \perp \cancel{V_4}$,

Embedding of one F_4 -building in another

Embed points T of F_4 -building with parameters (s', t') in this building:
(inner distribution \mathbf{a} of T consists just of valencies in smallest building)

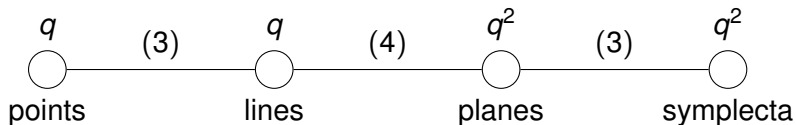
- if $(s', t') = (q, 1)$, $(s, t) = (q, q)$, then
 $\chi_T \in V_0 \perp V_1 \perp V_2 \perp \cancel{V_3} \perp \cancel{V_4}$,
- if $(s', t') = (q, q)$, $(s, t) = (q, q^2)$, then
 $\chi_T \in V_0 \perp V_1 \perp \cancel{V_2} \perp \cancel{V_3} \perp \cancel{V_4}$,

Embedding of one F_4 -building in another

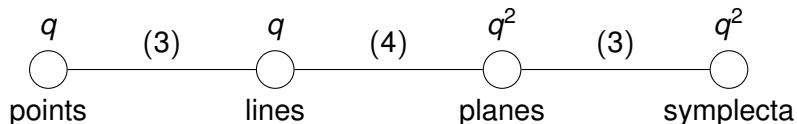
Embed points T of F_4 -building with parameters (s', t') in this building:
(inner distribution \mathbf{a} of T consists just of valencies in smallest building)

- if $(s', t') = (q, 1)$, $(s, t) = (q, q)$, then
 $\chi_T \in V_0 \perp V_1 \perp V_2 \perp \cancel{V_3} \perp \cancel{V_4}$,
- if $(s', t') = (q, q)$, $(s, t) = (q, q^2)$, then
 $\chi_T \in V_0 \perp V_1 \perp \cancel{V_2} \perp \cancel{V_3} \perp \cancel{V_4}$,
- if $(s', t') = (q, q)$, $(s, t) = (q^2, q)$, then
 $\chi_T \in V_0 \perp V_1 \perp \cancel{V_2} \perp V_3 \perp \cancel{V_4}$,

An immediate consequence of the previous for $(s, t) = (q, q^2)$!

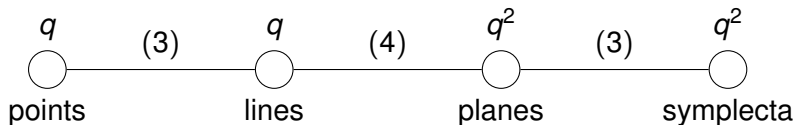


An immediate consequence of the previous for $(s, t) = (q, q^2)$!



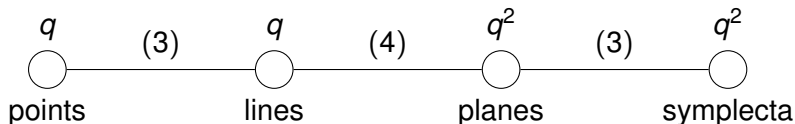
- S : set of $1 + s^3 t^3 = 1 + q^9$ mutually opposite points
 $\implies \chi_S \in V_0 \perp \cancel{V_1} \perp V_2 \perp V_3 \perp V_4$,

An immediate consequence of the previous for $(s, t) = (q, q^2)$!



- S : set of $1 + s^3 t^3 = 1 + q^9$ mutually opposite points
 $\implies \chi_S \in V_0 \perp \cancel{V_1} \perp V_2 \perp V_3 \perp V_4$,
- T : point set of embedded F_4 -building with $(s', t') = (q, q)$:
 $\implies \chi_T \in V_0 \perp V_1 \perp \cancel{V_2} \perp \cancel{V_3} \perp \cancel{V_4}$.

An immediate consequence of the previous for $(s, t) = (q, q^2)$!



- S : set of $1 + s^3 t^3 = 1 + q^9$ mutually opposite points
 $\implies \chi_S \in V_0 \perp \cancel{V_1} \perp V_2 \perp V_3 \perp V_4$,
- T : point set of embedded F_4 -building with $(s', t') = (q, q)$:
 $\implies \chi_T \in V_0 \perp V_1 \perp \cancel{V_2} \perp \cancel{V_3} \perp \cancel{V_4}$.

Design-orthogonality yields: $|S \cap T| = \langle \chi_S, \chi_T \rangle = \frac{|S||T|}{|\Omega|} = q^3 + 1$.

Thank you for your attention!