A geometric proof of the upper bound on the size of partial spreads in $H(4n+1, q^2)$

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Outline

- The problem and its history
- Algebraic approach
- Geometric approach

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- There are six types of finite nonsingular classical polar spaces:

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$H(2n,q^2)$	$+1/2$	$n-1$
$Q(2n,q)$	0	$n-1$
$W(2n+1,q)$	0	n
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- *Generators* in a polar space $\mathcal{P}$= subspaces of maximal projective dimension d.
- Number of points in $\mathcal{P}$: $(q^{d+1+\epsilon}+1)(q^{d+1}-1)/(q-1)$.

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- When does a polar space $\mathcal{P}$ have a spread?
- If it doesn't, what size can its partial spreads still have?

Hermitian variety $H(n, q^2)$

Our main focus is on the Hermitian variety $H(n, q^2)$ (and in particular $H(2n+1, q^2)$):
Subspaces in $\mathrm{PG}(n, q^2)$, all points $(X_0, \ldots, X_n)$ of which satisfy:
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Properties of $H(2n+1, q^2)$

- Maximal projective dimension of subspaces $d = n$, and $\epsilon = -1/2$.
- Ovoid number (and hence upper bound on size partial spreads): $q^{2n+1} + 1$.

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New result

The maximum size of a partial spread in $H(4n+1, q^2)$ is $q^{2n+1}+1$.

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Dual polar graph Γ on generators (or d-spaces) of polar space $\mathcal{P}$: two generators π_1, π_2 are adjacent iff $dim(\pi_1 \cap \pi_2) = d - 1$.

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- i-distance graph Γ_i: two generators are connected iff they meet in a $(d - i)$-space.
- Two generators are at maximal distance $d + 1$, and hence connected in Γ_{d+1}, iff they are disjoint.

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(In particular, R can be any i-distance graph Γ_i of a distance-regular graph Γ.)

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Remark

For any other polar space (except in a silly case) $1 - k/\lambda_{min}$ is just $q^{d+1+\epsilon}+1$ (=the ovoid number)....

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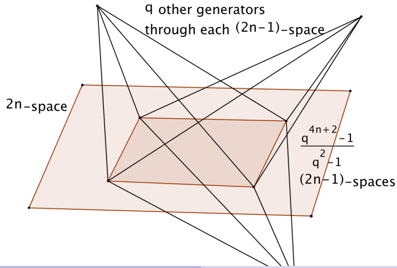
- There are $q+1$ generators through a $(2n-1)$-space in $H(4n+1, q^2)$.
- Hence every generator has $(q^{4n+2}-1)/(q^2-1)q$ neighbours.



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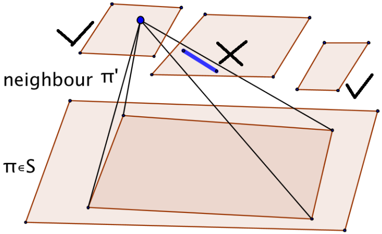
- A neighbour π' of $\pi \in S$ meets every other element of S in at most one point.



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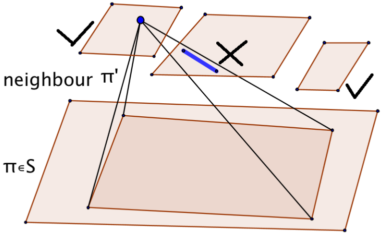

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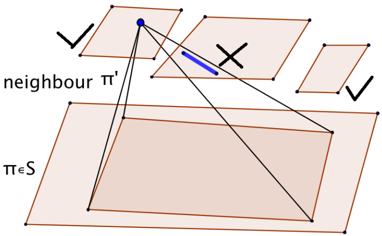

- In particular, a generator is neighbour of at most one element of S!
- The clue: consider the number of covered points on neighbours!

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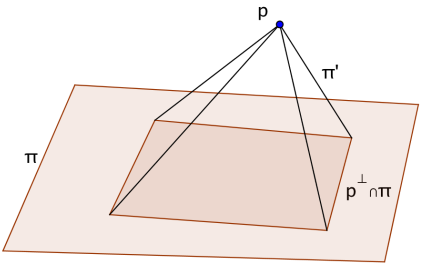
There is exactly one neighbour π' of π through p.



Consider 3 mutually disjoint generators (or $2n$-spaces) π, π_1, π_2 and any point $p \in \pi_1$ in $H(4n+1, q^2)$.

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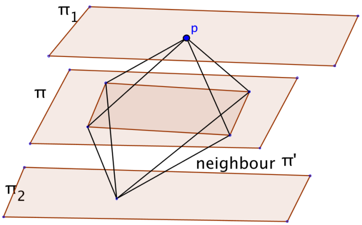
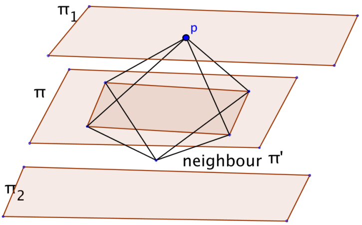
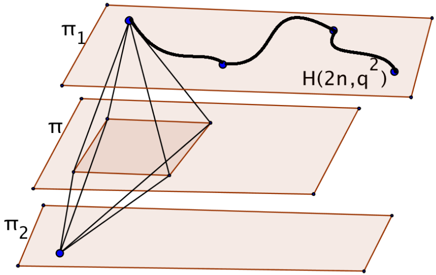




Lemma (Thas, 1990)

If π, π_1 and π_2 are mutually disjoint generators in $H(4n+1, q^2)$, the points on π_1, on neighbours of π also meeting π_2, form a non-singular Hermitian variety $H(2n, q^2)$ within π_1.



Theorem

Let S be a partial spread in $H(4n+1, q^2)$, with $\pi \in S$ (and $|S| > 1$). Then :

- $|S| \leq q^{2n+1} + 1$,
- $|S| = q^{2n+1} + 1$ iff every neighbour of π contains the same number of covered points (not in π),

That constant must also be q^{2n}.

Proof

- Let $N = \{N_i | i \in I\}$ be the set of neighbours of $\pi \in S$.
- $\forall N_i$: let t_i be the number of elements of S meeting N_i in a point (and hence the number of covered points of $N_i \backslash \pi$).

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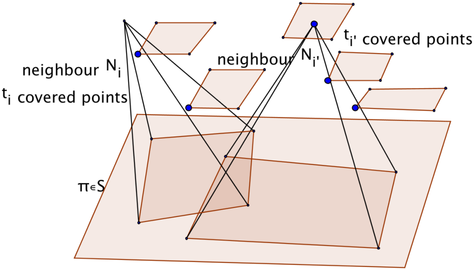

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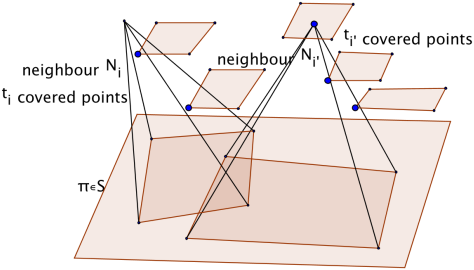

$$\sum_{i \in I} 1 = |I| = (q^{4n+2} - 1)/(q^2 - 1)q.$$

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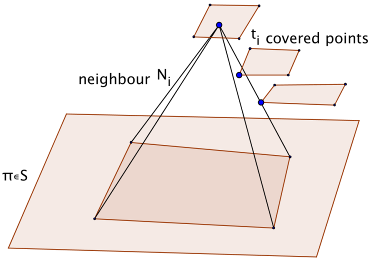

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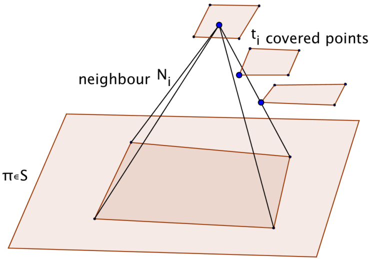

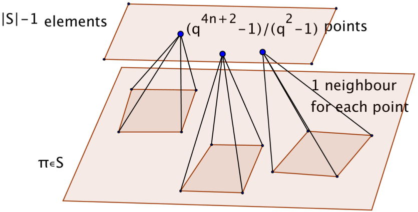

$$\sum_{i \in I} t_i = (|S| - 1)\frac{q^{4n+2} - 1}{q^2 - 1}.$$

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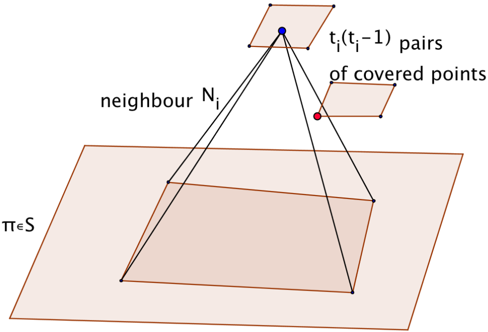
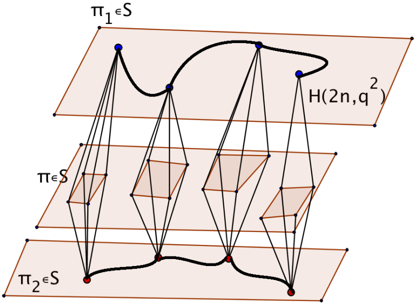
Counting ordered triples (N_i, p_1, p_2), with p_1, p_2 covered points in $N_i \backslash \pi$:
Method 1: for every neighbour N_i, there are $t_i(t_i - 1)$ possibilities for (p_1, p_2).



Number of triples (method 1): $\sum_{i \in I} t_i(t_i - 1)$.

Counting ordered triples (N_i, p_1, p_2), with p_1, p_2 covered points in $N_i \backslash \pi$: Method 2: for each of the $(|S|-1)(|S|-2)$ pairs (π_1, π_2) of elements of $S \backslash \{\pi\}$, there are $|H(2n, q^2)|$ neighbours of π meeting π_1 and π_2.



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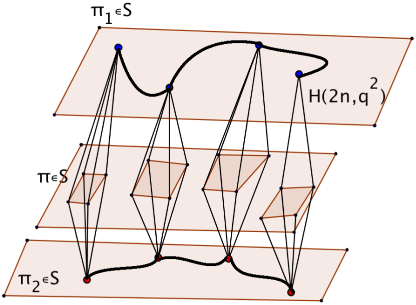

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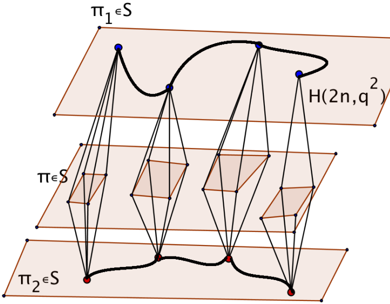

Number of triples (method 2): $(|S|-1)(|S|-2)||H(2n, q^2)|$.
Hence: $\sum_{i \in I} t_i(t_i - 1) = (|S|-1)(|S|-2)(q^{2n+1}+1)(q^{2n}-1)/(q^2-1)$.

We already obtained:

$$\sum_{i \in I} 1 = \frac{q^{4n+2}-1}{q^2-1} q,$$

$$\sum_{i \in I} t_i = (|S|-1)\frac{q^{4n+2}-1}{q^2-1},$$

$$\begin{aligned}\sum_{i \in I} t_i^2 &= \sum_{i \in I} t_i + \sum_{i \in I} t_i(t_i-1) \\ &= (|S|-1)\frac{q^{2n+1}+1}{q^2-1}\Big((q^{2n+1}-1)+(|S|-2)(q^{2n}-1)\Big).\end{aligned}$$

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$(\sum_{i\in I} t_i)^2 \leq (\sum_{i\in I} t_i^2)(\sum_{i\in I} 1)$ with equality iff all t_i are equal:

$$(|S|-1)^2\left(\frac{q^{4n+2}-1}{q^2-1}\right)^2 \leq (|S|-1)\frac{q^{2n+1}+1}{q^2-1}\left((q^{2n+1}-1)+(|S|-2)(q^{2n}-1)\right)\frac{q^{4n+2}-1}{q^2-1}q,$$

$(\sum_{i \in I} t_i)^2 \leq (\sum_{i \in I} t_i^2)(\sum_{i \in I} 1)$ with equality iff all t_i are equal:

$$(|S|-1)^2\left(\frac{q^{4n+2}-1}{q^2-1}\right)^2 \leq (|S|-1)\frac{q^{2n+1}+1}{q^2-1}\left((q^{2n+1}-1)+(|S|-2)(q^{2n}-1)\right)\frac{q^{4n+2}-1}{q^2-1}q,$$

or (as $|S| > 1$):

$$|S| \leq q^{2n+1}+1,$$

with equality iff all t_i are equal.

$(\sum_{i\in I} t_i)^2 \leq (\sum_{i\in I} t_i^2)(\sum_{i\in I} 1)$ with equality iff all t_i are equal:

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or (as $|S|>1$):

$$|S| \leq q^{2n+1}+1,$$

with equality iff all t_i are equal.

If equality is reached, then t_i (the number of covered points on $N_i\backslash\pi$) is always:

$$\sum_{i\in I} t_i/|I| = (|S|-1)/q = q^{2n}.$$

□

Thank you for your attention!