A geometric proof of the upper bound on the size of partial spreads in $H(4n+1, q^2)$

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Outline

- The problem and its history
- Algebraic approach
- Geometric approach

Our goal is to study partial spreads in polar spaces.

- There are six types of finite nonsingular classical polar spaces:

$\mathcal{P}$	ϵ	d
$Q^-(2n+1,q)$	$+1$	$n-1$
$H(2n,q^2)$	$+1/2$	$n-1$
$Q(2n,q)$	0	$n-1$
$W(2n+1,q)$	0	n
$H(2n+1,q^2)$	$-1/2$	n
$Q^+(2n+1,q)$	-1	n

- *Generators* in a polar space $\mathcal{P}$= subspaces of maximal projective dimension d.
- Number of points in $\mathcal{P}$: $(q^{d+1+\epsilon}+1)(q^{d+1}-1)/(q-1)$.

(Partial) Spreads

- A *partial spread* in a polar space $\mathcal{P}$ is a set of pairwise disjoint generators (or subspaces of maximal dimension d).
- The size of a partial spread in $\mathcal{P}$ is obviously at most: $|\mathcal{P}|/\frac{q^{d+1}-1}{q-1} = q^{d+1+\epsilon} + 1$ (=the *ovoid number* of $\mathcal{P}$).
- A *spread* of $\mathcal{P}$ is a partition of $\mathcal{P}$ into generators, or hence a partial spread of size $q^{d+1+\epsilon} + 1$.

Questions

- When does a polar space $\mathcal{P}$ have a spread?
- If it doesn't, what size can its partial spreads still have?

Hermitian variety $H(n, q^2)$

Our main focus is on the Hermitian variety $H(n, q^2)$ (and in particular $H(2n+1, q^2)$):
Subspaces in $\mathrm{PG}(n, q^2)$, all points $(X_0, \ldots, X_n)$ of which satisfy:
$X_0^{q+1} + \cdots + X_n^{q+1} = 0$.

Properties of $H(2n+1, q^2)$

- Maximal projective dimension of subspaces $d = n$, and $\epsilon = -1/2$.
- Ovoid number (and hence upper bound on size partial spreads): $q^{2n+1} + 1$.

What is already known on partial spreads in $H(2n+1, q^2)$?

- J. A. Thas (1990): $H(2n+1, q^2)$ has no spreads, or hence no partial spreads of size $q^{2n+1}+1$.
- Aguglia, Cossidente & Ebert (2001): construction of partial spreads of size $q^{n+1}+1$ in $H(2n+1, q^2)$ (and in particular of size $q^{2n+1}+1$ in $H(4n+1, q^2)$).
- De Beule & Metsch (2007): the maximum size of a partial spread in $H(5, q^2)$ is q^3+1.

New result

The maximum size of a partial spread in $H(4n+1, q^2)$ is $q^{2n+1}+1$.

Definition

Dual polar graph Γ on generators (or d-spaces) of polar space $\mathcal{P}$: two generators π_1, π_2 are adjacent iff $dim(\pi_1 \cap \pi_2) = d - 1$.

Properties and notations

- Γ is a distance-regular graph with diameter $d + 1$.
- $d(\pi_1, \pi_2) = i$ if and only if $\pi_1 \cap \pi_2$ is a $(d - i)$-space.
- i-distance graph Γ_i: two generators are connected iff they meet in a $(d - i)$-space.
- Two generators are at maximal distance $d + 1$, and hence connected in Γ_{d+1}, iff they are disjoint.

Cliques

If R is a symmetric relation on Ω, a subset $S \subseteq \Omega$ is a *clique* of R if $(x, y) \in R, \forall x \neq y \in S$.

Theorem

If R is any relation of an association scheme $\mathcal{R}$ on Ω, with valency k and eigenvalue $\lambda < 0$, then for any clique S of R: $|S| \leq 1 - k/\lambda$.
(In particular, R can be any i-distance graph Γ_i of a distance-regular graph Γ.)

Partial spreads as cliques

The partial spreads of a polar space $\mathcal{P}$ are the cliques of Γ_{d+1}, with Γ the dual polar graph.

Theorem

The size of a partial spread in $H(4n+1, q^2)$ is at most $q^{2n+1}+1$.

Proof:

Consider the disjointness-relation Γ_{d+1} between generators in $H(4n+1, q^2)$, with $d = 2n$:

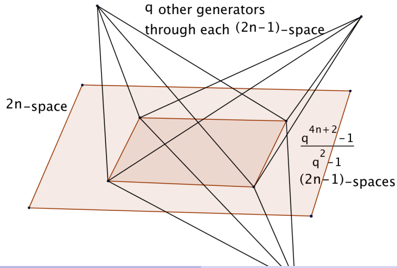
- Valency $k = q^{(2n+1)^2}$.
- Minimal eigenvalue $\lambda = -q^{2n(2n+1)}$.

If S is a partial spread (or clique): $|S| \leq 1 - k/\lambda = 1 + q^{2n+1}$. □

Remark

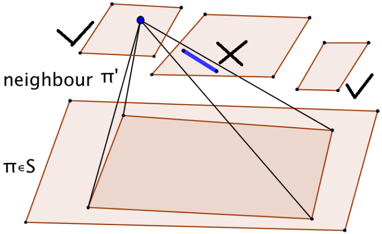
For any other polar space (except in a silly case) $1 - k/\lambda_{min}$ is just $q^{d+1+\epsilon}+1$ (=the ovoid number)....

- Two generators (or $2n$-spaces) in $H(4n+1, q^2)$ are *neighbours* if they meet in a $(2n-1)$-space.
- Each $2n$-space contains $(q^{4n+2}-1)/(q^2-1)$ $(2n-1)$-spaces.
- There are $q+1$ generators through a $(2n-1)$-space in $H(4n+1, q^2)$.
- Hence every generator has $(q^{4n+2}-1)/(q^2-1)q$ neighbours.



Let S be a partial spread of $H(4n+1, q^2)$:

- A point is *covered* if it is in a (unique) element of S.
- A neighbour π' of $\pi \in S$ meets every other element of S in at most one point.

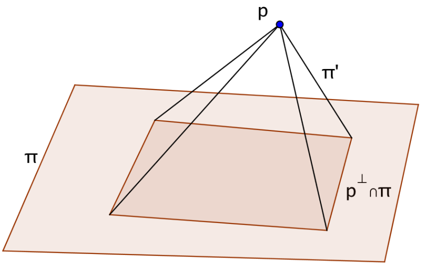


- In particular, a generator is neighbour of at most one element of S!
- The clue: consider the number of covered points on neighbours!

Consider:

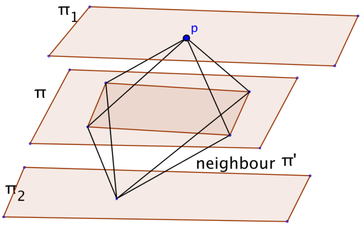
- any generator or $2n$-space π in $H(4n+1, q^2)$,
- any point p of $H(4n+1, q^2)$, not on π.

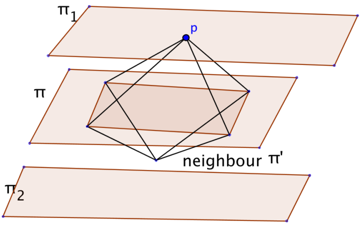
There is exactly one neighbour π' of π through p.



Consider 3 mutually disjoint generators (or $2n$-spaces) π, π_1, π_2 and any point $p \in \pi_1$ in $H(4n+1, q^2)$.
The unique neighbour π' of π through p either:

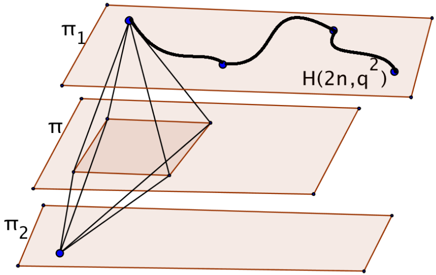
- meets π_2 in a unique point,
- or is disjoint from π_2.





Lemma (Thas, 1990)

If π, π_1 and π_2 are mutually disjoint generators in $H(4n+1, q^2)$, the points on π_1, on neighbours of π also meeting π_2, form a non-singular Hermitian variety $H(2n, q^2)$ within π_1.



Theorem

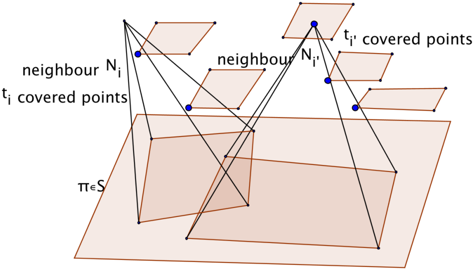
Let S be a partial spread in $H(4n+1, q^2)$, with $\pi \in S$ (and $|S| > 1$). Then :

- $|S| \leq q^{2n+1} + 1$,
- $|S| = q^{2n+1} + 1$ iff every neighbour of π contains the same number of covered points (not in π),

That constant must also be q^{2n}.

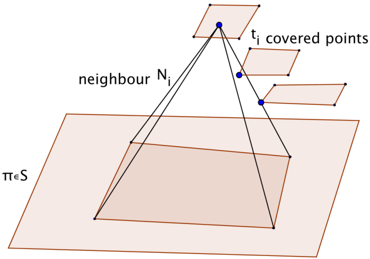
Proof

- Let $N = \{N_i | i \in I\}$ be the set of neighbours of $\pi \in S$.
- $\forall N_i$: let t_i be the number of elements of S meeting N_i in a point (and hence the number of covered points of $N_i \backslash \pi$).



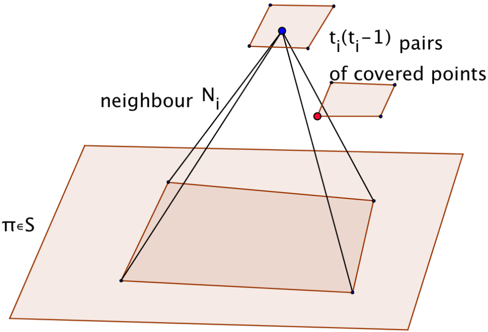
$$\sum_{i \in I} 1 = |I| = (q^{4n+2} - 1)/(q^2 - 1)q.$$

Counting pairs (N_i, p), with p a covered point in $N_i \backslash \pi$ in two ways:



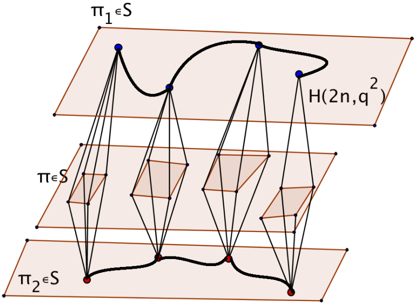
$$\sum_{i \in I} t_i = (|S| - 1)\frac{q^{4n+2} - 1}{q^2 - 1}.$$

Counting ordered triples (N_i, p_1, p_2), with p_1, p_2 covered points in $N_i \backslash \pi$:
Method 1: for every neighbour N_i, there are $t_i(t_i - 1)$ possibilities for (p_1, p_2).



Number of triples (method 1): $\sum_{i \in I} t_i(t_i - 1)$.

Counting ordered triples (N_i, p_1, p_2), with p_1, p_2 covered points in $N_i \backslash \pi$:
Method 2: for each of the $(|S|-1)(|S|-2)$ pairs (π_1, π_2) of elements of $S \backslash \{\pi\}$, there are $|H(2n, q^2)|$ neighbours of π meeting π_1 and π_2.



Number of triples (method 2): $(|S|-1)(|S|-2)||H(2n, q^2)|$.
Hence: $\sum_{i \in I} t_i(t_i - 1) = (|S|-1)(|S|-2)(q^{2n+1}+1)(q^{2n}-1)/(q^2-1)$.

We already obtained:

$$\begin{aligned}
\sum_{i\in I} 1 &= \frac{q^{4n+2}-1}{q^2-1}q, \\
\sum_{i\in I} t_i &= (|S|-1)\frac{q^{4n+2}-1}{q^2-1}, \\
\sum_{i\in I} t_i^2 &= \sum_{i\in I} t_i + \sum_{i\in I} t_i(t_i-1) \\
&= (|S|-1)\frac{q^{2n+1}+1}{q^2-1}\left((q^{2n+1}-1)+(|S|-2)(q^{2n}-1)\right).
\end{aligned}$$

$(\sum_{i\in I} t_i)^2 \leq (\sum_{i\in I} t_i^2)(\sum_{i\in I} 1)$ with equality iff all t_i are equal:

$$(|S|-1)^2\left(\frac{q^{4n+2}-1}{q^2-1}\right)^2 \leq (|S|-1)\frac{q^{2n+1}+1}{q^2-1}\left((q^{2n+1}-1)+(|S|-2)(q^{2n}-1)\right)\frac{q^{4n+2}-1}{q^2-1}q,$$

or (as $|S|>1$):

$$|S| \leq q^{2n+1}+1,$$

with equality iff all t_i are equal.

If equality is reached, then t_i (the number of covered points on $N_i\backslash\pi$) is always:

$$\sum_{i\in I} t_i/|I| = (|S|-1)/q = q^{2n}.$$

□

Thank you for your attention!