Erdős-Ko-Rado problems in polar spaces

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(joint work with Valentina Pepe and Leo Storme)

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Outline

- Introduction of EKR problem in polar spaces
- Approach using algebraic graph theory
- Exceptional cases
- Extras: Open problems & other ideas

Constructing classical polar spaces

- Consider $V(n, q)$ and a non-singular quadratic, alternating or Hermitian form f.
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Particular types of objects

- *points*: totally isotropic 1-spaces
- *lines*: totally isotropic 2-spaces
- *maximals*: totally isotropic d-spaces

Types of classical finite polar spaces

Polar space has parameters $(s,t) = (s, s^e)$ if:

- every line contains $s+1$ points
- every t.i. $(d-1)$-space is in exactly $t+1$ maximals

		(s,t)	e
$Q^+(2d-1,q)$	$D_d(q)$	$(q,1)$	0
$H(2d-1,q^2)$	${}^2A_{2d-1}(q)$	(q^2,q)	1/2
$Q(2d,q)$	$B_d(q)$	(q,q)	1
$W(2d-1,q)$	$C_d(q)$	(q,q)	1
$H(2d,q^2)$	${}^2A_{2d}(q)$	(q^2,q^3)	3/2
$Q^-(2d+1,q)$	${}^2D_{d+1}(q)$	(q,q^2)	2

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Good candidate

Point-pencil: set of maximals through fixed isotropic 1-space (=point) !

Related graphs

- Original EKR problem for subsets $\Longrightarrow$ Johnson graph
- EKR for subspaces $\Longrightarrow$ Grassmann graph
- EKR for polar spaces $\Longrightarrow$ dual polar graph!

Dual polar graph

Consider polar space of rank d with parameters (q, q^e):

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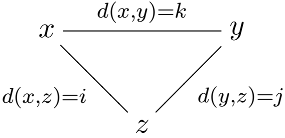
Some properties

- number of vertices: $(q^e+1)\cdots(q^{d-1+e}+1)$, valency: $q^e\left(\frac{q^d-1}{q-1}\right)$
- two d-spaces are at distance $i \iff \dim(\pi \cap \pi') = d - i$
- Γ has diameter d and is *distance-regular*: if $d(x,y) = k$ then # z with $d(x,z) = i, d(y,z) = j$ is constant p_{ij}^k:



Some properties (continued)

Consider polar space of rank d with parameters (q, q^e):

- Maximal clique of dual polar graph =
  all $q^e + 1$ maximals through fixed $(d-1)$-space

Observations on maximal EKR set S

Consider polar space of rank d with parameters (q, q^e):

- Each maximal clique has $0, 1$ or all its $q^e + 1$ elements in S (external, tangent or secant $(d-1)$-spaces)

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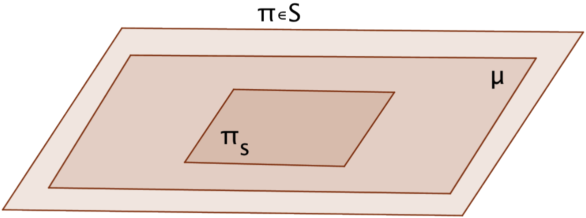

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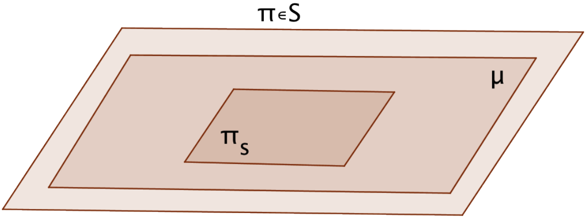

- $\pi \in S$ then has exactly $q^e\left(\frac{q^{d-s}-1}{q-1}\right)$ neighbours in S.

Consider a polar space of rank d with parameters (q, q^e) , with set of maximals Ω.

- For every $i \in \{0, \ldots, d\}$: *adjacency matrix* A_i is $(0,1)$-matrix with $(A_i)_{\pi_1,\pi_2} = 1 \iff d(\pi_1, \pi_2) = i$, $(A_i)_{\pi_1,\pi_2} = 0$ if not.

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$$q^e\left(\frac{q^{d-j}-1}{q-1}\right) - \frac{q^j-1}{q-1} \text{ for } V_j.$$

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- For a subset $S \subseteq \Omega$: *characteristic vector* χ_S:

$$\chi_S = (1,1,0,\dots,1,0,1)^T,$$

with $(\chi_S)_\omega = 1$ if $\omega \in S$, $(\chi_S)_\omega = 0$ if $\omega \notin S$.

In polar space of rank d:
EKR set of maximals S
= set of pairwise non-trivially intersecting maximals
= set of vertices in dual polar graph Γ, no two at distance d
= cocliques of maximum distance relation w.r.t Γ

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For most types of polar spaces:

- Upper bound = size of point-pencil EKR set
- Equality $\Longrightarrow \chi_S \in (V_0 \perp V_1)$

Consider a non-empty subset S in any distance-regular graph Γ.

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EKR sets of maximals = subsets in dual polar graph with $w < d$!

Known results

- Brouwer-Godsil-Koolen-Martin (2003):
  subsets with $w + w^* = d$ yield induced subschemes
- Tanaka (2006):
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Some immediate consequences

In most polar spaces, if S has width $w \leq d - 1$:

- $|S| \leq$ size point-pencil construction,
- equality $\Longrightarrow \chi_S \in V_0 \perp V_1$ (i.e. dual width $w^* = 1$),

$\Longrightarrow$ EKR sets of maximum size = point-pencils !

Hyperbolic quadric $D_d(q)/Q^+(2d-1,q)$ for even d

- Upper bound for EKR set $S = 2(q+1)\cdots(q^{d-2}+1)$ $=$ size point-pencil construction
- equality $\Longrightarrow \chi_S \in V_0 \perp V_1 \perp V_{d-1}$
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- set of vertices: one half
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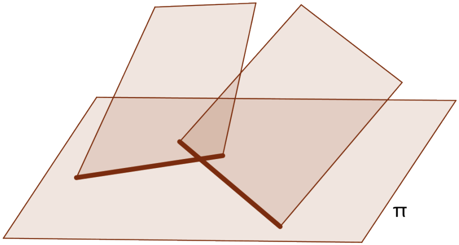
New approach

- We look for EKR sets of size $(q+1)\cdots(q^{d-2}+1)$ in each half.
- Here they satisfy $w + w^* = d'$ with $w = 1$ and $w^* = d' - 1$.

EKR sets of maximum size in one half of $D_d(q)/Q^+(2d-1,q)$

- $\forall \pi \in S$: we can count those in S intersecting π in a 2-space (=line)
- Using $w + w^* =$ diameter $\Longrightarrow$ the 2-spaces intersect non-trivially, and there are at least $\frac{(q^{d-1}-1)(q^{d-2}-1)}{(q^2-1)(q-1)}$ such lines:



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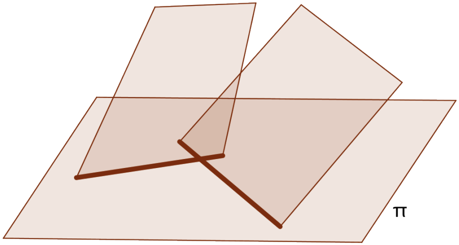

- Erdős-Ko-Rado for vector space $V(d,q)$: for $d \geq 6$: they are the lines through fixed 1-space (=point)

The symplectic space $C_d(q)/W(2d-1,q)$ for odd d

- Upper bound for $|S|$: size of a point-pencil but...
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Approach

- Eigenspace V_d = kernel incidence matrix between $(d-1)$-spaces and d-spaces
- $\Longrightarrow$ counting elements in S w.r.t $(d-1)$-spaces is easier
- similar ideas by Calderbank-Delsarte (1993) and Delsarte (2004)

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- Recall: every $(d-1)$-space has $0, 1$ or $q+1$ of the d-spaces through it in the maximal EKR set (external, tangent or secant)
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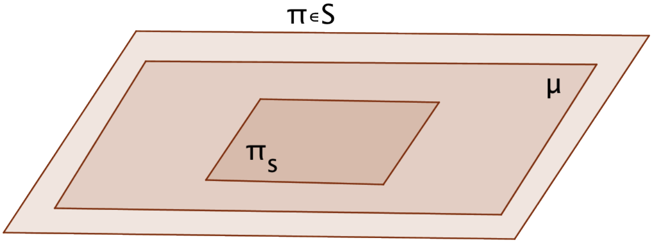

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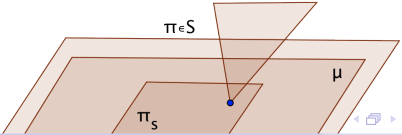

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... or: $0 \leq s \leq 2$.

Classification of EKR sets
(i.e. subsets of maximal totally isotropic subspaces,
pairwise intersecting non-trivially)
of maximum size in all polar spaces...
...except for $^2A_{2d-1}(q)/H(2d-1,q^2)$ for odd $d \geq 5$.

${}^2A_{2d-1}(q)/H(2d-1,q^2)$ for odd d

- size point-pencil construction: $|\Omega|/(q^{2d-1}+1)$
- Hoffman bound: EKR set S satisfies $|S| \leq |\Omega|/(q^d+1)$, with equality iff $\chi_S \in V_0 \perp V_d$

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Small rank d

- $d = 3$: EKR set of maximum size: one 3-space + all those intersecting it in line (1-sphere in graph)

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- $d=5$:
  size point-pencil $\sim q^{16}$,
  Delsarte's linear programming bound: $\sim q^{17}$,
  Hoffman bound: $\sim q^{20}$

Alternative approach to symplectic $C_d(q)/W(2d-1,q)$ for odd d

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  same vertices as dual polar graph $C_d(q)/W(2d-1,q)$,
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- EKR set S of maximum size:
  sets with $w^* = 1$ and $w + w^* =$ diameter in Ustimenko graph
- Tanaka (2010): classified all sets with $w + w^* =$ diameter
  in 15 families of graphs, including Ustimenko graphs

Alternative approach to symplectic $C_d(q)/W(2d-1,q)$ for odd d?

- From parameters $B_d(q)/Q(2d,q)$ or $C_d(q)/W(2d-1,q)$:
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Linear programming bound

- Usually much higher than known constructions!
- In some cases only integer for few q!

Thank you for your attention!

Slides (and more) on `http://cage.ugent.be/~fvanhove`