Hemisystem-like constructions of classical distance-regular graphs

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June 25, 2011, Bled

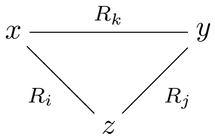
Outline

- Introduction:
  association schemes, distance-regular graphs, classical parameters
- Construction methods for schemes
- Properties/characterizations of hemisystem-like construction
- Another scheme
- Feasibility hemisystem-like construction?

Association schemes

$(\Omega, \{R_0, \ldots, R_d\})$, with $\Omega \neq \emptyset$ finite set, is an association scheme if:

- $\{R_0, \ldots, R_d\}$ partitions $\Omega \times \Omega$,
- R_0 is identity relation,
- $(\omega_1, \omega_2) \in R_i \iff (\omega_2, \omega_1) \in R_i$,
- there are *intersection numbers* p_{ij}^k:
  if $(x, y) \in R_k$, the number of elements z in Ω
  for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is p_{ij}^k.



Definition of matrices A_i

Consider association scheme $(\Omega, \{R_0, \ldots, R_d\})$
and order the elements of Ω: $\omega_1, \ldots, \omega_{|\Omega|}$.
For each R_i, define real $(|\Omega| \times |\Omega|)$-matrix A_i:

$$\begin{cases} (A_i)_{rs} = 1 \text{ if } (\omega_r, \omega_s) \in R_i, \\ (A_i)_{rs} = 0 \text{ if } (\omega_r, \omega_s) \notin R_i. \end{cases}$$

Properties

- A_0 is identity matrix.
- $A_0 + \ldots + A_d$ is all-one matrix.
- A_i is symmetric.
- $A_i A_j = \sum_k p_{ij}^k A_k$.

Bose-Mesner algebra: algebra with basis $\{A_0, \ldots, A_d\}$.

Eigenvectors for scheme $(\Omega, \{R_0, \ldots, R_d\})$

$\mathbb{R}^\Omega$ is real vector space with basis indexed by elements of Ω.
$\mathbb{R}^\Omega$ uniquely decomposes as:

$$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d,$$

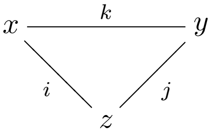
with every $v \neq 0 \in V_j$ an eigenvector of every relation R_i: $A_i v = \lambda_{ji} v$.

Idempotents

- Orthogonal projection E_j onto V_j is also in $\langle A_0, \ldots, A_d \rangle$.
- These *minimal idempotents* form second basis $\{E_0, \ldots, E_d\}$.

Distance-regular graphs

- Consider graph $\Gamma = (\Omega, E)$ of diameter d.
- Write $\Gamma_i(x)$ for vertices at distance i from x.
- Define R_i as $\{(x, y) \in \Omega \times \Omega | d(x, y) = i\}$.
- Γ is *distance-regular* if $(\Omega, \{R_0, \ldots, R_d\})$ is association scheme.



So number of z in $\Gamma_i(x) \cap \Gamma_j(y)$ is constant p_{ij}^k.

Convention:

$$c_k = |\Gamma_1(x) \cap \Gamma_{k-1}(y)|, a_k = |\Gamma_1(x) \cap \Gamma_k(y)|, b_k = |\Gamma_1(x) \cap \Gamma_{k+1}(y)|.$$

Such association schemes are called *P-polynomial* or *metric*.

Q-polynomial association schemes

- Bose-Mesner algebra $\langle A_0, \ldots, A_d \rangle = \langle E_0, \ldots, E_d \rangle$ is also closed under entrywise multiplication "$\circ$".
- $E_0, \ldots, E_d$ is a *Q-polynomial* or *cometric* ordering if:

$$|\Omega|(E_1 \circ E_i) = b^*_{i-1} E_{i-1} + a^*_i E_i + c^*_{i+1} E_{i+1},$$

with $c^*_i \neq 0$ for $i \in \{1, \ldots, d\}$.

Classical parameters

- Problem by Bannai-Ito (1984):
  find all P- and Q-polynomial schemes (for large d)
- Brouwer-Cohen-Neumaier (1989):
  many such schemes have *classical parameters* (d, b, α, β):

$$
\begin{aligned}
b_i = p^i_{1,i+1} &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right), \\
c_i = p^i_{1,i-1} &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right),
\end{aligned}
$$

  with $\begin{bmatrix} i \\ 1 \end{bmatrix}_b = 1 + \ldots + b^{i-1}$.
- b must be an integer $\notin \{-1, 0\}$.
- Can we find all those schemes (or graphs) with $b < -1$?

Main idea

- Consider a “sufficiently nice association scheme”
- Find a very special subset of vertices.
- Use this to create new association scheme!

Special subsets in scheme $(\Omega, \{R_0, \ldots, R_d\})$

- Characteristic vector of subset $S \subseteq \Omega$:

$$\chi_S = (1, 1, 0, \ldots, 1, 0)^T \in \mathbb{R}^\Omega,$$

  with $(\chi_S)_\omega = 1$ if $\omega \in S$ and $(\chi_S)_\omega = 0$ if not.
- Important observation: for $S_1, S_2 \subseteq \Omega : (\chi_{S_1})^T \chi_{S_2} = |S_1 \cap S_2|$.
- Recall:$E_0, \ldots, E_d$ are orthogonal projections onto eigenspaces for every A_i.
- Delsarte theory: subsets are "special" if $E_j(\chi_S) = 0$ for many j.

Nice schemes

- Recall: algebra $\langle A_0, \ldots, A_d \rangle = \langle E_0, \ldots, E_d \rangle$ is closed under $\circ$.
- Q-polynomial ordering $E_0, \ldots, E_d$ is *(almost) dual bipartite* if:

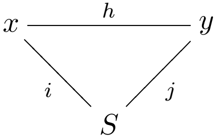
$$|\Omega|(E_1 \circ E_i) = b^*_{i-1}E_{i-1} + c^*_{i+1}E_{i+1}, \forall i \in \{1, \ldots, d-1\},$$

i.e. if $a^*_1 = a^*_2 = \ldots = a^*_{d-1} = 0$.

- Terwilliger (1988): in that case, $\forall (x, y) \in R_h \subseteq (\Omega \times \Omega)$:

$$E_1\chi_S = E_1(r^h_{ij}\chi_{\{x\}} + r^h_{ji}\chi_{\{y\}}),$$

with $S := \{s|(x, s) \in R_i, (y, s) \in R_j\}$.



Construction I : last subconstituents

- Suppose Γ is distance-regular, yielding (almost) dual bipartite scheme.
- Last subconstituent: $\Gamma_d(p)$ for some vertex p.
- If $x, y \in \Gamma_d(p)$, working out

$$(\chi_{\{p\}})^T E_1 \Big(\chi_S - (r_{ij}^h \chi_{\{x\}} + r_{ji}^h \chi_{\{y\}})\Big) = 0,$$

  yields *triple intersection numbers* and hence (often) distance-regular graph with vertex-set $\Gamma_d(p)$.
- Cameron-Goethals-Seidel (1978): strongly regular graphs ($d = 2$)

Construction II : regular partitions

- Suppose $(\Omega, \{R_0, \ldots, R_d\})$ is (almost) dual bipartite scheme.
- Suppose $S \subseteq \Omega$ satisfies $|S| = |\Omega|/2$ and $\chi_S = (E_0 + E_1)\chi_S$.
- If $x, y \in S$, working out

$$\left(\chi_S - \frac{1}{2}\chi_\Omega\right)^T \left(\chi_S - (r_{ij}^h \chi_{\{x\}} + r_{ji}^h \chi_{\{y\}})\right) = 0,$$

yields parameters of scheme with vertex-set S.

Construction II : Regular partitions (some examples)

- Higman-Sims graph srg$(100, 22, 0, 1)$ splits into Hoffman-Singleton graphs srg$(50, 7, 0, 1)$.
- 2nd subconstituent of McLaughlin graph, srg$(162, 56, 10, 24)$, splits into Brouwer-Haemers graphs srg$(81, 20, 1, 6)$.
- Removing points from (classical) generalized quadrangle $^2D_3(q)$ yields dual bipartite 4-class scheme.
  Penttila-Williford (2011): nice half “relative hemisystem” yields Q-polynomial 3-class scheme.

Unitary dual polar graph ${}^2A_{2d-1}(q)$

Consider non-deg. Hermitian form on $V(2d, q^2)$.

- Vertices ${}^2A_{2d-1}(q)$: totally isotropic d-dimensional spaces
- Adjacency: when intersecting in $(d-1)$-space

Properties

- Maximal cliques:
  $q+1$ d-spaces through totally isotropic $(d-1)$-space
- Number of vertices: $(q+1)(q^3+1)\dots(q^{2d-1}+1)$
- Classical parameters $(d, b, \alpha, \beta) = (d, q^2, 0, q)$ and
  $(d, b, \alpha, \beta) = (d, -q, -q(q+1)/(q-1), -q((-q)^d+1)/(q-1))$.
  Ivanov-Shpectorov (1989): characterized by parameters for $d \geq 3$.
- Almost dual bipartite (w.r.t. "2nd Q-polynomial ordering")

Construction I : last subconstituent

Let Γ be ${}^2A_{2d-1}(q)$.

- $\Gamma_d(p)$= set of vertices at distance d from p
  =set of totally isotropic d-spaces intersecting d trivially
- Induced graph on $\Gamma_d(p)$ is *Hermitian forms graph* $\mathrm{Her}(d,q)$.
- $\mathrm{Her}(d,q)$ has classical parameters $(d, -q, -q-1, -(-q)^d - 1)$.
  Ivanov-Shpectorov (1991): characterized by parameters for $d \geq 3$.

Construction II : hemisystem-like

Let Γ be ${}^2A_{2d-1}(q)$ with q odd.

- Suppose S is half of vertex set with $\chi_S = (E_0 + E_1)\chi_S$.
- Here: $\Longleftrightarrow$ every maximal clique of Γ has half its elements in S.
- S induces graph with classical parameters

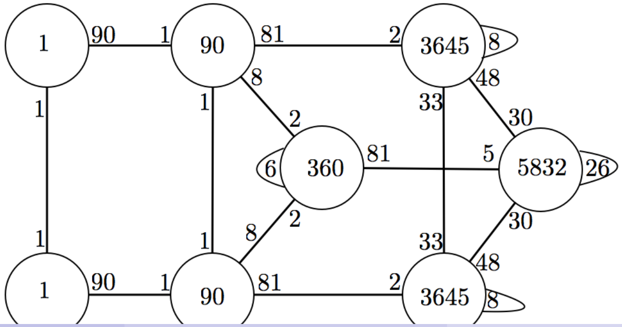
$$(d, -q, -(q+1)/2, -((-q)^d + 1)/2).$$

- Open for $d \geq 3$ (more to come)
- For $d = 2$: proved by Segre (1965), Cameron (1979), Thas (1981), halves are known as "hemisystems"

The case $q = 3$

Suppose Γ has classical parameters $(d, -q, -(q+1)/2, -((-q)^d + 1)/2)$.

- Since $a_1 = (q-3)/2$, Γ is triangle-free iff $q = 3$.
- For $d = 2, q = 3$, Gewirtz graph srg$(56, 10, 0, 2)$ is only possibility.
- Miklavič (2004) :
  any Q-polynomial triangle-free graph is 1-*homogeneous*.
- Example: $d = 3, q = 3$: equitable partition for any 2 neighbours:



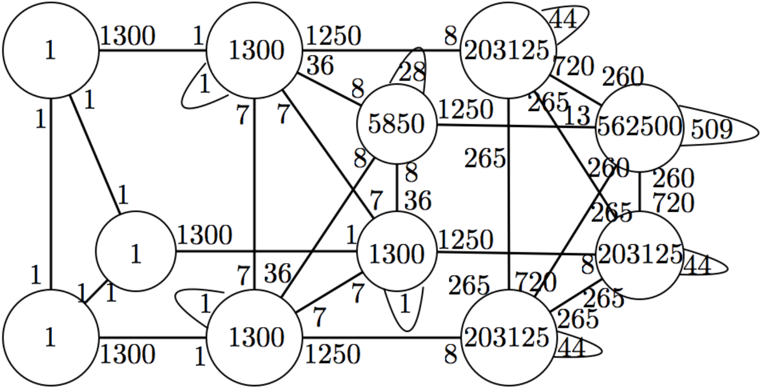
The case $q \geq 5$ for classical parameters $(d, -q, -(q+1)/2, -((-q)^d+1)/2)$

Here: $a_1 = (q-3)/2 > 0$ and $c_2 = (q-1)^2/2 > 1$.
Weng (1999):
If Γ has classical parameters (d, b, α, β) with $b < -1$ and $d \geq 4$ and $a_1 > 0, c_2 > 1$:

1. $\Gamma \cong$ unitary dual polar graph ${}^2A_{2d-1}(q)$,
2. $\Gamma \cong \mathrm{Her}(d, q)$ (Construction I: last subconstituent of ${}^2A_{2d-1}(q)$)
3. q is odd prime power and $(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d+1)/2)$ (none known for $d \geq 3$) (as in Construction II: special half of ${}^2A_{2d-1}(q)$)

The case $q \geq 5$ for classical parameters $(d, -q, -(q+1)/2, -((-q)^d+1)/2)$

- Miklavič (2005): "almost 1-homogeneous": for any 2 neighbours: equitable partition of graph into $4d-1$ cells
- Example: $d = 3$ and $q = 5$:



Using both halves

- Suppose Γ unitary dual polar graph ${}^2A_{2d-1}(q)$ with vertex set Ω.
- If S_1 is half satisfying $\chi_{S_1} = (E_0 + E_1)\chi_{S_1}$, then so is $S_2 = \Omega \backslash \Omega_1$.
- If $d(x,y) = i$ in Γ, let $(x,y) \in R_i^+$ if x, y in same half, in R_i^- if not.
- Terwilliger's property (almost) dual bipartiteness $\Longrightarrow$ $(\Omega, \{R_0, R_1^+, \ldots, R_d^+, R_1^-, \ldots, R_d^-\})$ is $2d$-class scheme!
- For $d = 2$: Martin-Muzychuk-Van Dam (2010)

Example: $d = 3$

- 6-class scheme on $(q+1)(q^3+1)(q^5+1)$ vertices
- Q-polynomial with $b_i^* = c_{d-i}^*$ (Q-antipodal).
- $b_i^* + c_{i+1}^* = b_0^* + 1, \forall i \in \{0, \dots, d-1\}$,
  Martin-Muzychuk-Williford (2007): $\Longrightarrow$
  scheme has *extended Q-bipartite double*,
  which is $(2d+1)$-class scheme

Necessary ingredient S for hemisystem-like Construction II

- Vertex set unitary dual polar graph ${}^2A_{2d-1}(q)$, q odd
  totally isotropic d-spaces $V(2d, q^2)$ w.r.t non-deg. Hermitian form
- half $S \subseteq \Omega$ satisfies $\chi_S = (E_0 + E_1)\chi_S \Longleftrightarrow$ every totally isotropic $(d-1)$-space in exactly $(q+1)/2$ elements of S
- Hence: we look for *designs in regular semilattices*
- For fixed q: existence for d implies existence for lower diameter!
- For $d = 2$: known as "hemisystems":
- Segre (1965): unique example for $d = 2$, $q = 3$
- Cossidente-Penttila (2005): existence for $d = 2$, q odd prime power
- Bamberg-Giudici-Royle (2011): more constructions for $d = 2$
- Cossidente-Penttila (2009): similar halving for $d = 3$

Thank you for your attention!

Slides (and more) on `http://cage.ugent.be/~fvanhove`