On geometry and distance-regular graphs

Frédéric Vanhove
UGent (Belgium)
fvanhove@cage.ugent.be, http://cage.ugent.be/~fvanhove

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Outline

- Introduction: distance-regular graphs, examples, eigenvalues,...
- Codes: inner distribution, designs, bounds,...
- Classification of (Q-polynomial) distance-regular graphs

Consider graph $\Gamma = (\Omega, E)$
(Ω: vertex set, E: set of edges (pairs of vertices))
($\Omega \neq \emptyset$, undirected, no loops or multiple edges)

- A *path of length* k is a sequence $(x_0, \ldots, x_k)$ with every 2 successive vertices adjacent.
- *Distance* between 2 vertices x, y: length of shortest path $(x, \ldots, y)$. (denoted by $d(x, y)$).
- *Diameter:* maximum distance in graph.
- $\Gamma_i(x)$: set of vertices at distance i from vertex x.

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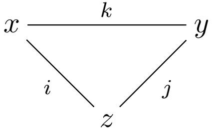
- *Diameter:* maximum distance in graph.
- $\Gamma_i(x)$: set of vertices at distance i from vertex x.

Distance-regular graphs (or drgs)

A connected graph $\Gamma = (\Omega, E)$ is *distance-regular* if for $d(x, y) = k$ number of z with $d(x, z) = i, d(y, z) = j$ is parameter p^k_{ij} only depending on i, j, k.



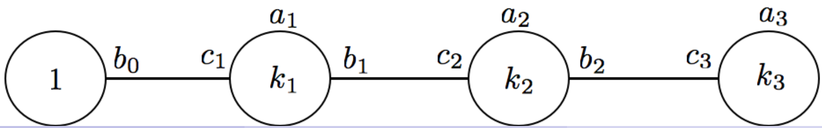
Distance-regularity is equivalent to:
There are constants a_i, b_i, c_i, such that for $d(x,y) = i$

$$|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i, |\Gamma_i(x) \cap \Gamma_1(y)| = a_i,$$
$$|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i.$$

Some properties

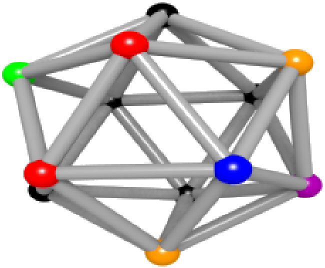
- $c_1, \ldots, c_d$ and $b_0, \ldots, b_{d-1}$ already determine all parameters p^k_{ij},
- number of vertices at distance i from any vertex is constant $k_i = (b_0 \cdots b_{i-1})/(c_1 \cdots c_i), \forall i \in \{1, \ldots, i\}$.

Distribution diagram



Example: icosahedron

12 vertices, diameter $d = 3$, every vertex has 5 neighbours.

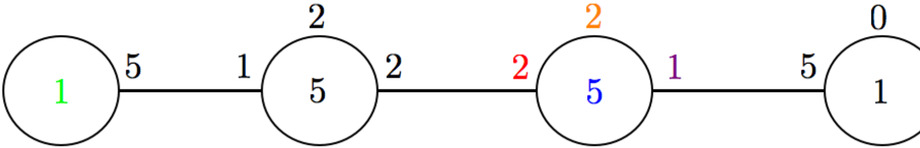


if $d(x, y) = 2$:

$\Gamma_1(x) \cap \Gamma_1(y) = c_2 = 2$

$\Gamma_2(x) \cap \Gamma_1(y) = a_2 = 2$

$\Gamma_3(x) \cap \Gamma_1(y) = b_2 = 1$.



Example: Johnson graph $J(n,d)$ (design theory)

Consider set X of size n, with $n \geq 2d$.

- vertices: all $\binom{n}{d}$ subsets of size d in X,
- adjacency: when intersection has size $d-1$.

Properties of $J(n,d)$

- distance $i \iff$ intersection has size $d-i$.
- $J(n,d)$ is distance-regular with $c_i = i^2,\ b_i = (d-i)(n-d-i)$.

Example: Grassmann graph $J_q(n,d)$ (projective geometry)

Consider vector space X of dimension n over GF(q), with $n \geq 2d$.

- vertices: all subspaces of dimension d in X,
- adjacency: when intersection has dimension $d-1$.

Properties of $J_q(n,d)$

- distance $i \iff$ intersection has dimension $d-i$.
- $J_q(n,d)$ is distance-regular with $c_i = \left(\frac{q^i-1}{q-1}\right)^2$, $b_i = q^{2i+1}\frac{(q^{d-i}-1)(q^{n-d-i}-1)}{(q-1)^2}$.
- For $q \to 1$, formulae for $J_q(n,d)$ become those for $J(n,d)$!

Example: Hamming graph $H(d,q)$ (coding theory)

- vertices: all q^d codewords of length d over alphabet of q symbols,
- adjacency: when differing in 1 position.

Properties of $H(d,q)$

- distance $i \iff$ codewords differ in i positions
- $H(d,q)$ is distance-regular with $c_i = i,\ b_i = (d-i)(q-1)$.

Example: Dual polar graphs (classical polar spaces)

Consider $V(n,q)$ with non-degenerate quadratic form/alternating/Hermitian form with Witt index d.

- vertices: all totally isotropic d-dimensional subspaces
- adjacency: when intersecting in $(d-1)$-dimensional space

Properties of dual polar graphs

- distance $i \iff$ intersection has dimension $d-i$.
- distance-regular with $c_i = \frac{q^i-1}{q-1}$, $b_i = q^{i+e}\left(\frac{q^{d-i}-1}{q-1}\right)$.
- For $q \to 1$, formulae become those for Hamming graph $H(d,2)$!

Adjacency matrices

Consider distance-regular graph Γ with vertex set Ω:
Adjacency matrix A_i: $(|\Omega| \times |\Omega|)$-matrix over $\mathbb{R}$:

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & \text{if } d(x,y) \neq i \end{cases} .$$

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- A_i is symmetric
- A_0 is identity matrix
- $A_0 + \cdots + A_d$ is all-one matrix
- $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$.

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$$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d,$$

with every $v \neq 0 \in V_j$ an eigenvector of every relation A_i: $A_i v = \lambda_{ji} v$.
P is $(d+1) \times (d+1)$-matrix with $P_{ji} = \lambda_{ji}$.

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Idempotents

- Orthogonal projection E_j onto V_j is also in $\langle A_0, \ldots, A_d \rangle$.
- These *minimal idempotents* E_j
  form second basis $\{E_0, \ldots, E_d\}$ for $\langle A_0, \ldots, A_d \rangle$.

Example: icosahedron

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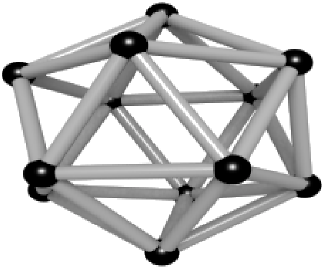
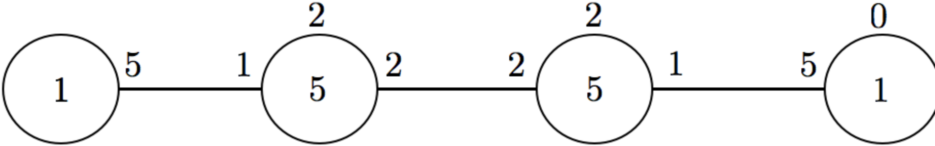


$$P = \begin{pmatrix} 1 & 5 & 5 & 1 \\ 1 & \sqrt{5} & -\sqrt{5} & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -\sqrt{5} & \sqrt{5} & -1 \end{pmatrix}$$



Q-matrix

- We write $Q := |\Omega| P^{-1}$
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Q-polynomial orderings

- Algebra $\langle A_0, \ldots, A_d \rangle$ is also closed under entrywise (!) multiplication.
- Γ has Q-polynomial ordering $E_0, E_1, \ldots, E_d$ if every E_j is polynomial of degree j in E_1 (entrywise!)
- Many graphs (Johnson, Hamming, Grassmann, dual polar graph) have Q-polynomial orderings (the "usual ordering" from semilattices)

Code in drg Γ = non-empty subset of vertices S

- *inner distribution* $\mathbf{a}$ of S:

$$\mathbf{a}_i = \frac{1}{|S|}|\{(x,y) \in (S \times S) \; |d(x,y) = i\}|.$$

  (so $\mathbf{a}_i$ is average valency of distance i within S)
- *MacWilliams transform*: $\mathbf{a}|\Omega|P^{-1} = \mathbf{a}Q$.
- *characteristic vector* χ_S is $(0,1)$-vector in $\mathbb{R}^\Omega$ with $(\chi_S)_\omega = 1$ if $\omega \in S$, $(\chi_S)_\omega = 0$ if $\omega \notin S$.

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Example: linear codes in $H(d,q)$

If q is prime power, S subspace of $(\mathbb{F}_q)^d$:

- $\mathbf{a}_i$: number of codewords in S of weight i,
- $\mathbf{a}Q$: $|S| \times$ inner distribution of dual code $S^\perp$.

$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d$.

- $\mathbf{a}_i = \frac{1}{|S|}(\chi_S)^T A_i \chi_S \geq 0$,
  with equality iff distance i does not occur within S
- $(\mathbf{a}Q)_j = \frac{|\Omega|}{|S|}(\chi_S)^T E_j \chi_S \geq 0$,
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Designs

Suppose Γ has Q-polynomial ordering $V_0, V_1, \ldots, V_d$.
$S \subseteq \Omega$ is *t-design* if $(\mathbf{a}Q)_1 = \ldots = (\mathbf{a}Q)_t = 0$.
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$S \subseteq \Omega$ is *t-antidesign* if $(\mathbf{a}Q)_{t+1} = \ldots = (\mathbf{a}Q)_d = 0$.

Designs in Johnson graph $J(n, d)$ (w.r.t. usual Q-pol. ordering)

- vertices: subsets size d in set of size $n \geq 2d$
- S is t-design $(0 \leq t \leq d) \iff$
  every subset of size t in equally many elements of S

Designs in Grassmann graph $J_q(n, d)$ (w.r.t. usual Q-pol. ordering)

- vertices: d-spaces in $V(n, q), n \geq 2d$
- S is t-design $(0 \leq t \leq d) \iff$
  every t-space in equally many elements of S

Designs in Hamming graph $H(d, q)$ (w.r.t. usual Q-pol. ordering)

- vertices: codewords of length d over alphabet size q
- S is t-design $(0 \leq t \leq d) \iff$
  every incomplete word with t entries in equally many elements of S

Designs in dual polar graphs (w.r.t. usual Q-pol. ordering)

- vertices: totally isotropic subspaces in vector space of max. dim. d
- S is t-design $(0 \leq t \leq d)$ (in usual Q-polynomial ordering) $\iff$
  every totally isotropic t-space in equally many elements of S

Some open problems:

Codes in $H(d, q)$ with minimal distance δ.

- We want large codes S with minimal distance at least δ, i.e. every two codewords differ in at least δ positions.
- This means that inner distribution satisfies $\mathbf{a}_1 = \ldots = \mathbf{a}_{\delta-1} = 0$.
- Maximum size of such S is denoted by $A_q(d, \delta)$.
- Linear Programmming (LP) bound helps...
- ... but what is precise value for $A_q(d, \delta)$?
- Schrijver (2005) and Gijswijt-Schrijver-Tanaka (2006): better bounds for $A_q(d, \delta)$ from Semidefinite Programming (SDP)

Some open problems:

Tight designs in $J(n, d)$

- If $S \neq \emptyset$ is t-design with $n \geq 2d \geq 2t$, then $|S| \geq \binom{n}{\lfloor t/2 \rfloor}$.
  S is a *tight t-design* in case of equality, and then t is even.
- Tight t-designs are nonempty t-designs
  with precisely $t/2$ distances in $J(n, d)$.
- Can $t \geq 6$?

Partial spreads in dual polar graph ${}^2A_{2d-1}(q)/H(2d-1,q^2)$

Vertices: totally isotropic d-spaces in $V(2d,q^2)$
w.r.t. non-degenerate Hermitian form.
$S \neq \emptyset$ is *partial spread* $\iff$
every 2 distinct elements of S intersect trivially
$\iff$ every two distinct elements of S at distance d in dual polar graph
$\iff$ S is code with $\mathbf{a} = (1, 0, \ldots, 0, |S| - 1)$.

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Theorem (V., 2009)(using Linear Programming Bound)

If d is odd: $|S| \leq q^d + 1$,
equality iff χ_S orthogonal to eigenspace V_d
for eigenvalue $-(q^{2d}-1)/(q^2-1)$ of A_1.
(i.e. iff S is $(d-1)$-antidesign w.r.t. usual Q-polynomial ordering)
(this bound is tight)

Consider dual polar graph ${}^2A_{2d-1}(q)/H(2d-1,q^2)$.

Designs

Segre (1965):
if T is non-trivial $(d-1)$-design w.r.t. usual Q-pol. ordering,
i.e. every totally isotropic $(d-1)$-space in λ elements of T,
then $\lambda = (q+1)/2$.
(for $d = 2$, such designs are *hemisystems*)

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Antidesigns

For odd d: partial spread S of size q^d+1 is $(d-1)$-antidesign.

Applying design-orthogonality (Delsarte (1977))

For odd d: $|S \cap T| = |S|/2 = (q^d+1)/2$.

Intersecting antidesigns in Q-polynomial drgs in general

Martin (2001):
t_1-antidesign and t_2-antidesign intersect in $(t_1 + t_2)$-antidesign

Intersecting $(d-1)$-designs T_1, T_2 in ${}^2A_{2d-1}(q)/H(2d-1, q^2)$

If T_1, T_2 are (hemisystem-like) $(d-1)$-designs in usual Q-pol. ordering:
T_1, T_2 are 1-antidesigns in 2nd Q-polynomial ordering.
$\Longrightarrow T_1 \cap T_2$ is 2-antidesign in 2nd ordering
$\Longrightarrow T_1 \cap T_2$ has $\mathbf{a}Q = (?, ?, 0, \ldots, 0, ?)$ w.r.t. usual ordering

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The case $d = 3$

- If totally isotropic 2-space ℓ in ${}^2A_5(q)/H(5, q^2)$ in x elements of $T_1 \cap T_2$,
  then $x(q^3 - 1) + |T_1 \cap T_2|/(q^3 + 1)$ elements of $T_1 \cap T_2$ intersect ℓ in 1-space.
- Similar ideas: Calderbank-Delsarte (1993):
  Extending the t-design concept.

Recall:

- Γ is distance-regular if $|\Gamma_i(x) \cap \Gamma_j(y)|$ for $d(x, y) = k$ only depends on k, i, j.
- drg Γ is Q-polynomial if there is (at least one) meaningful ordering $E_0, E_1, \ldots, E_d$ for $d + 1$ eigenspaces.
- The b_i and c_i completely determine parameters, eigenvalues, P- and Q-matrix, Q-polynomiality,...

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Bannai-Ito (1984): find all Q-polynomial drgs (for large diameter d).
(this includes Johnson, Hamming and Grassmann graphs,
dual polar graphs, Ustimenko graphs, bilinear forms graphs,
generalized hexagons of order $(s, s^3)(s \geq 2)$,...)

Grassmann graph $J_q(n,d)$, $n \geq 2d$:

Properties

- vertices: d-subspaces in $V(n,q)$
  adjacency: when intersection has dimension $d-1$.
- Q-polynomial drg with $c_i = \left(\frac{q^i-1}{q-1}\right)^2$, $b_i = q^{2i+1}\frac{(q^{d-i}-1)(q^{n-d-i}-1)}{(q-1)^2}$.

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Characterization of $J_q(n,d)$ by parameters?

Metsch (1995): if $d \geq 3, q \geq 4$ and $n \geq 2d+2$,
then $J_q(n,d)$ is only graph with those parameters.

Grassmann graph $J_q(2d+1,d)$

- vertices: d-subspaces in $V(2d+1,q)$
- adjacency: when intersection has dimension $d-1$.
- Q-pol. drg with $c_i = \left(\frac{q^i-1}{q-1}\right)^2$, $b_i = q^{2i+1}\frac{(q^{d-i}-1)(q^{d+1-i}-1)}{(q-1)^2}$.
- $\mathrm{Aut}(J_q(2d+1,d))$ acts transitively on pairs with fixed distance.

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Twisted Grassmann graph $\tilde{J}_q(2d+1,d)$ (van Dam-Koolen (2005))

Consider hyperplane H in $V(2d+1,q)$

- vertices: type I: $(d+1)$-spaces not in H,
  type II: $(d-1)$-spaces in H
- adjacency: same type: intersecting in hyperplane;
  different type: strict inclusion.
- Q-pol. drg. with $c_i = \left(\frac{q^i-1}{q-1}\right)^2$, $b_i = q^{2i+1}\frac{(q^{d-i}-1)(q^{d+1-i}-1)}{(q-1)^2}$.
- $\mathrm{Aut}(\tilde{J}_q(2d+1,d))$ has 2 orbits on vertices!

Classical parameters

Brouwer-Cohen-Neumaier (1989):
drg Γ has *classical parameters* (d, b, α, β) if:

$$\begin{aligned} b_i &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right), \\ c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b \right), \end{aligned}$$

with $\begin{bmatrix} i \\ 1 \end{bmatrix}_b = 1 + \ldots + b^{i-1}$.

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Properties

- If $d \geq 3$: b is integer $\notin \{-1, 0\}$, and α, β are rational.
- Γ has Q-polynomial ordering.

Examples

Most (but not all!) known Q-polynomial drgs have classical parameters (d, b, α, β):

		d	b	α	β
Johnson	$J(n,d)$	d	1	1	$n-d$
Grassmann	$J_q(n,d)$	d	q	q	$q\frac{q^{n-d}-1}{q-1}$
Hamming	$H(d,q)$	d	1	0	$q-1$
dual polar		d	q	0	q^e
unitary dual polar	${}^2A_{2d-1}(q)$	d	$-q$	$-q\frac{q+1}{q-1}$	$-q\frac{(-q)^d+1}{q-1}$
Hermitian forms	$\mathrm{Her}(d,q)$	d	$-q$	$-q-1$	$-(-q)^d-1$
twisted triality hexagon	${}^3D_{4,2}(q)$	3	$-q$	$-\frac{q}{q-1}$	$q(q+1)$
exceptional Lie graph	$E_{7,7}(q)$	3	q^4	$q\frac{q^4-1}{q-1}$	$q\frac{q^9-1}{q-1}$

Classifying distance-regular graphs (in general)

1. Consider possibility of parameter sets $b_0, \ldots, b_{d-1}$ and $c_1, \ldots, c_d$.
2. Find graphs for certain parameter sets.
3. Prove all graphs are known for certain parameter sets.

Classifying distance-regular graphs (in general)

1. Consider possibility of parameter sets $b_0, \ldots, b_{d-1}$ and $c_1, \ldots, c_d$.
2. Find graphs for certain parameter sets.
3. Prove all graphs are known for certain parameter sets.

Some examples for Step 1

- Many results by Terwilliger
- De Bruyn (2010): non-existence for $(d, b, \alpha, \beta) = (4, -2, -2, -10)$
- De Bruyn-V. (201?):
  If Γ is point graph near $2d$-gon with $s = a_1 + 1 \geq 2$ and $i \geq 3$:

$$\frac{(s^i - 1)(c_{i-1} - s^{i-2})}{s^{i-2} - 1} \leq c_i \leq \frac{(s^i + 1)(c_{i-1} + s^{i-2})}{s^{i-2} + 1}$$

- ...

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Some examples for Step 2

- Tits (1959): generalized hexagons with $c_2 = 1$ and $(a_1 + 1, c_3 - 1) \in \{(q, q), (q^3, q), (q, q^3)\}$, with q prime power
- Doob (1972): graphs with same parameters as $H(d, 4)$
- van Dam-Koolen (2005):
  graphs with same parameters as $J_q(2d + 1, d)$
- ....

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Some examples for Step 3

- Egawa (1981):
  characterization of Hamming/Doob graphs by parameters.
- Ivanov-Shpectorov (1991):
  characterization Hermitian forms graph $\mathrm{Her}(d, q), d \geq 3$ by classical parameters $(d, -q, -q-1, -(-q)^d - 1)$.
- Metsch (1995): characterization of $J_q(n, d)$ by parameters under certain conditions.

Classical parameters (d, b, α, β) with $b < -1$ (Step 1)

Weng (1999):
If Γ has classical parameters (d, b, α, β) with $b < -1$
and $d \geq 4$ and $a_1 > 0, c_2 > 1$:

1. $\Gamma \cong$ unitary dual polar graph ${}^2A_{2d-1}(q)/H(2d-1, q^2)$,
2. $\Gamma \cong \mathrm{Her}(d, q)$ (subgraph of ${}^2A_{2d-1}(q)/H(2d-1, q^2)$)
3. q is odd prime power and
$(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d + 1)/2)$
(none known for $d \geq 3$)

Theorem (V., 2011) (Step 2??)

Consider dual polar graph ${}^2A_{2d-1}(q)/H(2d-1,q^2)$
(from $V(2d,q^2)$ w.r.t. non-deg. Hermitian form).
If S is (hemisystem-like) $(d-1)$-design (set of totally isotropic d-spaces such that each $(d-1)$-space is in exactly $(q+1)/2$ elements of S),
$\Longrightarrow$ S yields subgraph with classical parameters (and thus Q-pol.)

$$(d,b,\alpha,\beta)=\left(d,-q,-\frac{q+1}{2},-\frac{(-q)^d+1}{2}\right).$$

Theorem (V., 2011) (Step 2??)

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Small d

- Segre (1965): result for $d = 2$, unique example for $q = 3$
- Cossidente-Penttila (2005): existence for all odd q if $d = 2$!
- Existence such designs for d implies it for lower diameters.

Selected references

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Thank you for your attention!

Slides (and more) on `http://cage.ugent.be/~fvanhove`