

On geometry and distance-regular graphs

Frédéric Vanhove
UGent (Belgium)

fvanhove@cage.ugent.be, <http://cage.ugent.be/~fvanhove>

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Outline

- Introduction: distance-regular graphs, examples, eigenvalues,...
- Codes: inner distribution, designs, bounds,...
- Classification of (Q -polynomial) distance-regular graphs

Consider graph $\Gamma = (\Omega, E)$

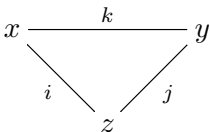
(Ω : vertex set, E : set of edges (pairs of vertices))

($\Omega \neq \emptyset$, undirected, no loops or multiple edges)

- A *path of length k* is a sequence (x_0, \dots, x_k) with every 2 successive vertices adjacent.
- *Distance* between 2 vertices x, y : length of shortest path (x, \dots, y) . (denoted by $d(x, y)$).
- *Diameter*: maximum distance in graph.
- $\Gamma_i(x)$: set of vertices at distance i from vertex x .

Distance-regular graphs (or drgs)

A connected graph $\Gamma = (\Omega, E)$ is *distance-regular* if for $d(x, y) = k$ number of z with $d(x, z) = i, d(y, z) = j$ is parameter p_{ij}^k only depending on i, j, k .



Distance-regularity is equivalent to:

There are constants a_i, b_i, c_i , such that for $d(x, y) = i$

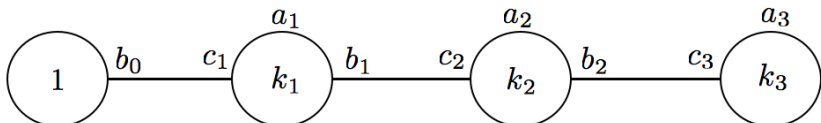
$$|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i, |\Gamma_i(x) \cap \Gamma_1(y)| = a_i,$$

$$|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i.$$

Some properties

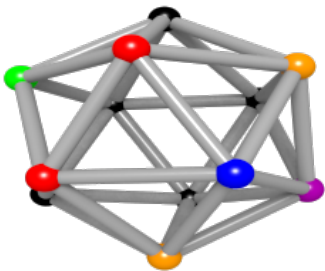
- c_1, \dots, c_d and b_0, \dots, b_{d-1} already determine all parameters p_{ij}^k ,
- number of vertices at distance i from any vertex is constant
 $k_i = (b_0 \cdots b_{i-1}) / (c_1 \cdots c_i), \forall i \in \{1, \dots, d\}$.

Distribution diagram



Example: icosahedron

12 vertices, diameter $d = 3$, every vertex has 5 neighbours.

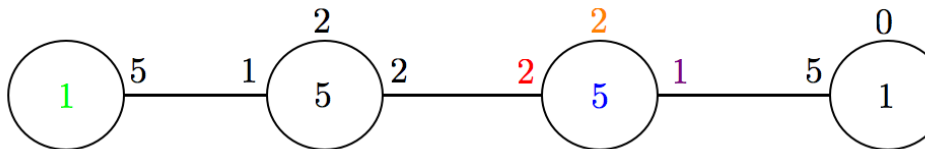


if $d(x, y) = 2$:

$$\Gamma_1(x) \cap \Gamma_1(y) = c_2 = 2$$

$$\Gamma_2(x) \cap \Gamma_1(y) = a_2 = 2$$

$$\Gamma_3(x) \cap \Gamma_1(y) = b_2 = 1.$$



Example: Johnson graph $J(n, d)$ (design theory)

Consider set X of size n , with $n \geq 2d$.

- vertices: all $\binom{n}{d}$ subsets of size d in X ,
- adjacency: when intersection has size $d - 1$.

Properties of $J(n, d)$

- distance $i \iff$ intersection has size $d - i$.
- $J(n, d)$ is distance-regular with $c_i = i^2$, $b_i = (d - i)(n - d - i)$.

Example: Grassmann graph $J_q(n, d)$ (projective geometry)

Consider vector space X of dimension n over $\text{GF}(q)$, with $n \geq 2d$.

- vertices: all subspaces of dimension d in X ,
- adjacency: when intersection has dimension $d - 1$.

Properties of $J_q(n, d)$

- distance $i \iff$ intersection has dimension $d - i$.
- $J_q(n, d)$ is distance-regular with

$$c_i = \left(\frac{q^i - 1}{q - 1}\right)^2, \quad b_i = q^{2i+1} \frac{(q^{d-i} - 1)(q^{n-d-i} - 1)}{(q - 1)^2}.$$
- For $q \rightarrow 1$, formulae for $J_q(n, d)$ become those for $J(n, d)$!

Example: Hamming graph $H(d, q)$ (coding theory)

- vertices: all q^d codewords of length d over alphabet of q symbols,
- adjacency: when differing in 1 position.

Properties of $H(d, q)$

- distance $i \iff$ codewords differ in i positions
- $H(d, q)$ is distance-regular with $c_i = i$, $b_i = (d - i)(q - 1)$.

Example: Dual polar graphs (classical polar spaces)

Consider $V(n, q)$ with non-degenerate quadratic form/alternating/Hermitian form with Witt index d .

- vertices: all totally isotropic d -dimensional subspaces
- adjacency: when intersecting in $(d - 1)$ -dimensional space

Properties of dual polar graphs

- distance $i \iff$ intersection has dimension $d - i$.
- distance-regular with $c_i = \frac{q^i - 1}{q - 1}$, $b_i = q^{i+1} \left(\frac{q^{d-i} - 1}{q - 1} \right)$.
- For $q \rightarrow 1$, formulae become those for Hamming graph $H(d, 2)$!

Adjacency matrices

Consider distance-regular graph Γ with vertex set Ω :

Adjacency matrix A_i : $(|\Omega| \times |\Omega|)$ -matrix over \mathbb{R} :

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & \text{if } d(x,y) \neq i \end{cases} .$$

- A_i is symmetric
- A_0 is identity matrix
- $A_0 + \cdots + A_d$ is all-one matrix
- $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$.

Eigenvectors for distance-regular graph Γ

\mathbb{R}^Ω is real vector space with basis indexed by elements of vertex set Ω .

\mathbb{R}^Ω uniquely decomposes as:

$$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \dots \perp V_d,$$

with every $v \neq 0 \in V_j$ an eigenvector of every relation A_i : $A_i v = \lambda_{ji} v$.

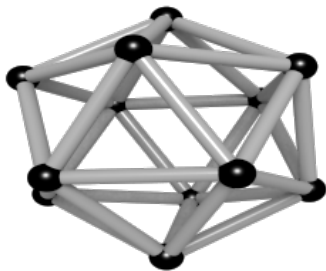
P is $(d+1) \times (d+1)$ -matrix with $P_{ji} = \lambda_{ji}$.

Idempotents

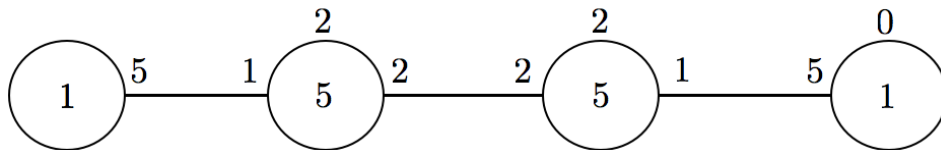
- Orthogonal projection E_j onto V_j is also in $\langle A_0, \dots, A_d \rangle$.
- These *minimal idempotents* E_j form second basis $\{E_0, \dots, E_d\}$ for $\langle A_0, \dots, A_d \rangle$.

Example: icosahedron

12 vertices, diameter $d = 3$, every vertex has 5 neighbours.



$$P = \begin{pmatrix} 1 & 5 & 5 & 1 \\ 1 & \sqrt{5} & -\sqrt{5} & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -\sqrt{5} & \sqrt{5} & -1 \end{pmatrix}$$



Q -matrix

- We write $Q := |\Omega|P^{-1}$
- j -th column of Q expresses $|\Omega|E_j$ as lin. comb. of A_i .

Q -polynomial orderings

- Algebra $\langle A_0, \dots, A_d \rangle$ is also closed under entrywise (!) multiplication.
- Γ has Q -polynomial ordering E_0, E_1, \dots, E_d if every E_j is polynomial of degree j in E_1 (entrywise!)
- Many graphs (Johnson, Hamming, Grassmann, dual polar graph) have Q -polynomial orderings (the “usual ordering” from semilattices)

Code in drg Γ = non-empty subset of vertices S

- *inner distribution* \mathbf{a} of S :

$$\mathbf{a}_i = \frac{1}{|S|} |\{(x, y) \in (S \times S) \mid d(x, y) = i\}|.$$

(so \mathbf{a}_i is average valency of distance i within S)

- *MacWilliams transform*: $\mathbf{a}|\Omega|P^{-1} = \mathbf{a}Q$.
- *characteristic vector* χ_S is $(0, 1)$ -vector in \mathbb{R}^Ω
with $(\chi_S)_\omega = 1$ if $\omega \in S$, $(\chi_S)_\omega = 0$ if $\omega \notin S$.

Example: linear codes in $H(d, q)$

If q is prime power, S subspace of $(\mathbb{F}_q)^d$:

- \mathbf{a}_i : number of codewords in S of weight i ,
- $\mathbf{a}Q$: $|S| \times$ inner distribution of dual code S^\perp .

$$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \dots \perp V_d.$$

- $\mathbf{a}_i = \frac{1}{|S|}(\chi_S)^T A_i \chi_S \geq 0$,
with equality iff distance i does not occur within S
- $(\mathbf{a}Q)_j = \frac{|\Omega|}{|S|}(\chi_S)^T E_j \chi_S \geq 0$,
with equality iff χ_S is orthogonal to V_j .
(this is Delsarte's *Linear Programming Bound* (1973))

Designs

Suppose Γ has Q -polynomial ordering V_0, V_1, \dots, V_d .

$S \subseteq \Omega$ is t -*design* if $(\mathbf{a}Q)_1 = \dots = (\mathbf{a}Q)_t = 0$.

(Delsarte (1976): often has nice combinatorial meaning!)

$S \subseteq \Omega$ is t -*antidesign* if $(\mathbf{a}Q)_{t+1} = \dots = (\mathbf{a}Q)_d = 0$.

Designs in Johnson graph $J(n, d)$ (w.r.t. usual Q -pol. ordering)

- vertices: subsets size d in set of size $n \geq 2d$
- S is t -design ($0 \leq t \leq d$) \iff
every subset of size t in equally many elements of S

Designs in Grassmann graph $J_q(n, d)$ (w.r.t. usual Q -pol. ordering)

- vertices: d -spaces in $V(n, q)$, $n \geq 2d$
- S is t -design ($0 \leq t \leq d$) \iff
every t -space in equally many elements of S

Designs in Hamming graph $H(d, q)$ (w.r.t. usual Q -pol. ordering)

- vertices: codewords of length d over alphabet size q
- S is t -design ($0 \leq t \leq d$) \iff
every incomplete word with t entries in equally many elements of S

Designs in dual polar graphs (w.r.t. usual Q -pol. ordering)

- vertices: totally isotropic subspaces in vector space of max. dim. d
- S is t -design ($0 \leq t \leq d$) (in usual Q -polynomial ordering) \iff
every totally isotropic t -space in equally many elements of S

Some open problems:

Codes in $H(d, q)$ with minimal distance δ .

- We want large codes S with minimal distance at least δ ,
i.e. every two codewords differ in at least δ positions.
- This means that inner distribution satisfies $\mathbf{a}_1 = \dots = \mathbf{a}_{\delta-1} = 0$.
- Maximum size of such S is denoted by $A_q(d, \delta)$.
- Linear Programming (LP) bound helps...
- ... but what is precise value for $A_q(d, \delta)$?
- Schrijver (2005) and Gijswijt-Schrijver-Tanaka (2006):
better bounds for $A_q(d, \delta)$ from Semidefinite Programming (SDP)

Some open problems:

Tight designs in $J(n, d)$

- If $S \neq \emptyset$ is t -design with $n \geq 2d \geq 2t$, then $|S| \geq \binom{n}{\lfloor t/2 \rfloor}$.
 S is a *tight t -design* in case of equality, and then t is even.
- Tight t -designs are nonempty t -designs with precisely $t/2$ distances in $J(n, d)$.
- Can $t \geq 6$?

Partial spreads in dual polar graph ${}^2A_{2d-1}(q)/H(2d-1, q^2)$

Vertices: totally isotropic d -spaces in $V(2d, q^2)$

w.r.t. non-degenerate Hermitian form.

$S \neq \emptyset$ is *partial spread* \iff

every 2 distinct elements of S intersect trivially

\iff every two distinct elements of S at distance d in dual polar graph

$\iff S$ is code with $\mathbf{a} = (1, 0, \dots, 0, |S| - 1)$.

Theorem (V., 2009)(using Linear Programming Bound)

If d is odd: $|S| \leq q^d + 1$,

equality iff χ_S orthogonal to eigenspace V_d

for eigenvalue $-(q^{2d} - 1)/(q^2 - 1)$ of A_1 .

(i.e. iff S is $(d-1)$ -antidesign w.r.t. usual Q -polynomial ordering)

(this bound is tight)

Consider dual polar graph ${}^2A_{2d-1}(q)/H(2d-1, q^2)$.

Designs

Segre (1965):

if T is non-trivial $(d-1)$ -design w.r.t. usual Q -pol. ordering,
i.e. every totally isotropic $(d-1)$ -space in λ elements of T ,
then $\lambda = (q+1)/2$.

(for $d=2$, such designs are *hemisystems*)

Antidesigns

For odd d : partial spread S of size $q^d + 1$ is $(d-1)$ -antidesign.

Applying design-orthogonality (Delsarte (1977))

For odd d : $|S \cap T| = |S|/2 = (q^d + 1)/2$.

Intersecting antidesigns in Q -polynomial drgs in general

Martin (2001):

 t_1 -antidesign and t_2 -antidesign intersect in $(t_1 + t_2)$ -antidesignIntersecting $(d - 1)$ -designs T_1, T_2 in ${}^2A_{2d-1}(q)/H(2d - 1, q^2)$ If T_1, T_2 are (hemisystem-like) $(d - 1)$ -designs in usual Q -pol. ordering:
 T_1, T_2 are 1-antidesigns in 2nd Q -polynomial ordering. $\implies T_1 \cap T_2$ is 2-antidesign in 2nd ordering $\implies T_1 \cap T_2$ has $\mathbf{a}Q = (?, ?, 0, \dots, 0, ?)$ w.r.t. usual orderingThe case $d = 3$

- If totally isotropic 2-space ℓ in ${}^2A_5(q)/H(5, q^2)$ in x elements of $T_1 \cap T_2$,
then $x(q^3 - 1) + |T_1 \cap T_2|/(q^3 + 1)$ elements of $T_1 \cap T_2$ intersect ℓ in 1-space.
- Similar ideas: Calderbank-Delsarte (1993):
Extending the t -design concept.

Recall:

- Γ is distance-regular if $|\Gamma_i(x) \cap \Gamma_j(y)|$ for $d(x, y) = k$ only depends on k, i, j .
- drg Γ is Q -polynomial if there is (at least one) meaningful ordering E_0, E_1, \dots, E_d for $d + 1$ eigenspaces.
- The b_i and c_i completely determine parameters, eigenvalues, P - and Q -matrix, Q -polynomiality,...

Bannai-Ito (1984): find all Q -polynomial drgs (for large diameter d).
(this includes Johnson, Hamming and Grassmann graphs,
dual polar graphs, Ustimenko graphs, bilinear forms graphs,
generalized hexagons of order $(s, s^3)(s \geq 2), \dots$)

Grassmann graph $J_q(n, d)$, $n \geq 2d$:

Properties

- vertices: d -subspaces in $V(n, q)$
adjacency: when intersection has dimension $d - 1$.
- Q -polynomial drg with $c_i = \left(\frac{q^i - 1}{q - 1}\right)^2$, $b_i = q^{2i+1} \frac{(q^{d-i} - 1)(q^{n-d-i} - 1)}{(q - 1)^2}$.

Characterization of $J_q(n, d)$ by parameters?

Metsch (1995): if $d \geq 3$, $q \geq 4$ and $n \geq 2d + 2$,
then $J_q(n, d)$ is only graph with those parameters.

Grassmann graph $J_q(2d+1, d)$

- vertices: d -subspaces in $V(2d+1, q)$
- adjacency: when intersection has dimension $d-1$.
- Q -pol. drg with $c_i = \left(\frac{q^i-1}{q-1}\right)^2$, $b_i = q^{2i+1} \frac{(q^{d-i}-1)(q^{d+1-i}-1)}{(q-1)^2}$.
- $\text{Aut}(J_q(2d+1, d))$ acts transitively on pairs with fixed distance.

Twisted Grassmann graph $\tilde{J}_q(2d+1, d)$ (van Dam-Koolen (2005))

Consider hyperplane H in $V(2d+1, q)$

- vertices: type I: $(d+1)$ -spaces not in H ,
type II: $(d-1)$ -spaces in H
- adjacency: same type: intersecting in hyperplane;
different type: strict inclusion.
- Q -pol. drg. with $c_i = \left(\frac{q^i-1}{q-1}\right)^2$, $b_i = q^{2i+1} \frac{(q^{d-i}-1)(q^{d+1-i}-1)}{(q-1)^2}$.
- $\text{Aut}(\tilde{J}_q(2d+1, d))$ has 2 orbits on vertices!

Classical parameters

Brouwer-Cohen-Neumaier (1989):

drg Γ has *classical parameters* (d, b, α, β) if:

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b \right),$$
$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b \right),$$

with $\begin{bmatrix} i \\ 1 \end{bmatrix}_b = 1 + \dots + b^{i-1}$.

Properties

- If $d \geq 3$: b is integer $\notin \{-1, 0\}$, and α, β are rational.
- Γ has Q -polynomial ordering.

Examples

Most (but not all!) known Q -polynomial drgs have classical parameters (d, b, α, β) :

		d	b	α	β
Johnson	$J(n, d)$	d	1	1	$n - d$
Grassmann	$J_q(n, d)$	d	q	q	$q \frac{q^{n-d}-1}{q-1}$
Hamming	$H(d, q)$	d	1	0	$q - 1$
dual polar		d	q	0	q^e
unitary dual polar	${}^2A_{2d-1}(q)$	d	$-q$	$-q \frac{q+1}{q-1}$	$-q \frac{(-q)^{d+1}}{q-1}$
Hermitian forms	$\text{Her}(d, q)$	d	$-q$	$-q - 1$	$-(-q)^d - 1$
twisted triality hexagon	${}^3D_{4,2}(q)$	3	$-q$	$-\frac{q}{q-1}$	$q(q+1)$
exceptional Lie graph	$E_{7,7}(q)$	3	q^4	$q \frac{q^4-1}{q-1}$	$q \frac{q^9-1}{q-1}$

Classifying distance-regular graphs (in general)

- 1 Consider possibility of parameter sets b_0, \dots, b_{d-1} and c_1, \dots, c_d .
- 2 Find graphs for certain parameter sets.
- 3 Prove all graphs are known for certain parameter sets.

Some examples for Step 1

- Many results by Terwilliger
- De Bruyn (2010): non-existence for $(d, b, \alpha, \beta) = (4, -2, -2, -10)$
- De Bruyn-V. (201?):

If Γ is point graph near $2d$ -gon with $s = a_1 + 1 \geq 2$ and $i \geq 3$:

$$\frac{(s^i - 1)(c_{i-1} - s^{i-2})}{s^{i-2} - 1} \leq c_i \leq \frac{(s^i + 1)(c_{i-1} + s^{i-2})}{s^{i-2} + 1}$$

- ...

Classifying distance-regular graphs (in general)

- 1 Consider possibility of parameter sets b_0, \dots, b_{d-1} and c_1, \dots, c_d .
- 2 Find graphs for certain parameter sets.
- 3 Prove all graphs are known for certain parameter sets.

Some examples for Step 2

- Tits (1959): generalized hexagons with $c_2 = 1$ and $(a_1 + 1, c_3 - 1) \in \{(q, q), (q^3, q), (q, q^3)\}$, with q prime power
- Doob (1972): graphs with same parameters as $H(d, 4)$
- van Dam-Koolen (2005): graphs with same parameters as $J_q(2d + 1, d)$
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Classifying distance-regular graphs (in general)

- 1 Consider possibility of parameter sets b_0, \dots, b_{d-1} and c_1, \dots, c_d .
- 2 Find graphs for certain parameter sets.
- 3 Prove all graphs are known for certain parameter sets.

Some examples for Step 3

- Egawa (1981):
characterization of Hamming/Doob graphs by parameters.
- Ivanov-Shpectorov (1991):
characterization Hermitian forms graph $\text{Her}(d, q)$, $d \geq 3$ by classical parameters $(d, -q, -q - 1, -(-q)^d - 1)$.
- Metsch (1995): characterization of $J_q(n, d)$ by parameters under certain conditions.

Classical parameters (d, b, α, β) with $b < -1$ (Step 1)

Weng (1999):

If Γ has classical parameters (d, b, α, β) with $b < -1$
and $d \geq 4$ and $a_1 > 0, c_2 > 1$:

- 1 $\Gamma \cong$ unitary dual polar graph ${}^2A_{2d-1}(q)/H(2d-1, q^2)$,
- 2 $\Gamma \cong \text{Her}(d, q)$ (subgraph of ${}^2A_{2d-1}(q)/H(2d-1, q^2)$)
- 3 q is odd prime power and
 $(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d + 1)/2)$
 (none known for $d \geq 3$)

Theorem (V., 2011) (Step 2??)

Consider dual polar graph ${}^2A_{2d-1}(q)/H(2d-1, q^2)$
(from $V(2d, q^2)$ w.r.t. non-deg. Hermitian form).

If S is (hemisystem-like) $(d-1)$ -design (set of totally isotropic d -spaces such that each $(d-1)$ -space is in exactly $(q+1)/2$ elements of S),
 $\implies S$ yields subgraph with classical parameters (and thus Q -pol.)

$$(d, b, \alpha, \beta) = \left(d, -q, -\frac{q+1}{2}, -\frac{(-q)^d + 1}{2} \right).$$

Small d

- Segre (1965): result for $d=2$, unique example for $q=3$
- Cossidente-Penttila (2005): existence for all odd q if $d=2$!
- Existence such designs for d implies it for lower diameters.

Selected references



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Thank you for your attention!

Slides (and more) on <http://cage.ugent.be/~fvanhove>