Eigenvalue techniques for regular and extremal substructures in geometry

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Outline

- Introduction: geometries, graphs
- Extremal (co)cliques: EKR
- Regular substructures: t-designs, hemisystems,....

Main idea

1. Consider a finite geometry (related to finite fields).
2. Associate graph(s) (with nice properties) with that geometry.
3. Use eigenvalues/eigenvectors of those graphs
   to consider substructures in geometry.

Example I: projective space from $V(n,q)$

- Objects are i-dimensional subspaces, with $i \in \{1,\ldots,n\}$.
- Incidence I is just symmetrized inclusion between subspaces.

Example II: classical finite polar spaces

Consider $V(n,q)$ with non-degenerate quadratic/alternating/Hermitian form.

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Other examples

Axiomatically defined geometries, often constructed using finite fields: projective planes, generalized quadrangles/hexagons/octagons,...

Graphs

A *graph* Γ consists of:

- a finite set $\Omega \neq \emptyset$: the *vertices*
- a set E of pairs of vertices: the *edges*.

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$x, y \in \Omega$ are *adjacent* or *neigbours* if $\{x, y\}$ is an edge of Γ.

We say Γ is *regular with valency k*
if every vertex has exactly k neighbours.

Eigenvalues and eigenvectors of graphs

Consider graph Γ with vertex set Ω:

- Adjacency matrix A: $(|\Omega| \times |\Omega|)$-matrix over $\mathbb{R}$:

$$(A)_{x,y} = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are adjacent} \\ 0 & \text{if } x \text{ and } y \text{ are not adjacent} \end{cases} .$$

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- *Eigenvalues of* Γ = the eigenvalues of A.
- A is real-symmetric $\Longrightarrow$ all eigenvalues are real.
- If Γ is regular with valency k then $A(1,1,\ldots,1)^T = k(1,1,\ldots,1)^T$, so k is eigenvalue (and largest eigenvalue).
- For any subset $S \subseteq \Omega$: *characteristic vector* $\chi_S = (1,1,\ldots,0,1)^T$ with 1 for elements of S, 0 for rest.

Cliques and cocliques

Consider any graph Γ with vertex set Ω.

- *Clique*: subset of vertices S, any two adjacent
- *Coclique*: subset of vertices S, no two adjacent

Question : how large can (co)cliques S in Γ be,
and what if they are precisely that large?

General theorem on cocliques (Hoffman)

Suppose Γ is regular with valency $k \geq 1$, vertex set Ω and adjacency matrix A.
Suppose $S \subseteq \Omega$ is a coclique.

- If λ_{min} is minimal eigenvalue of Γ then

$$|S| \leq \frac{|\Omega|}{1 - k/\lambda_{min}}.$$

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- In case of equality:
  χ_S is linear combination of eigenvectors of k and λ_{min}.

Original Erdős-Ko-Rado problem

- Erdős-Ko-Rado (1961): in a set X of size n, let S be a collection of subsets of size d ($n \geq 2d+1$). If no two subsets in S are disjoint, then $|S| \leq \binom{n-1}{d-1}$, with equality if and only if all elements of S contain the same element of X.

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  with equality if and only if
  all elements of S contain the same element of X.
- *Kneser graph* $K(n, d)$: vertices are subsets in X of size d,
  adjacent when intersection is empty.
- Hence we are studying cocliques in $K(n, d)$!

Erdős-Ko-Rado problem for vector spaces over finite fields

- Hsieh (1975): in a vector space $V(n,q)$,
  let S be a collection of subspaces of dimension d ($n \geq 2d+1$).
  If no two subspaces in S intersect trivially, then
  $|S| \leq \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q = \prod_{j=1}^{d-1} \frac{q^{n-j}-1}{q^j-1}$,
  with equality iff all elements of S contain same 1-space in $V(n,q)$.

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- *q-Kneser graph* $K_q(n, d)$:
  vertices are subspaces in $V(n, q)$ of dimension d,
  adjacent when intersection is trivial.
- Hence we are studying cocliques in $K_q(n, d)$!
- Improvements/alternative proofs: Green-Kleitman (1978),
  Frankl-Wilson (1986), Godsil-Newman (2006), Tanaka (2006)

Erdős-Ko-Rado problem for classical finite polar spaces

Consider finite-dimensional vector space
+ non-degen. quadratic/alternating/Hermitian form with Witt index d.
Polar space consists of totally isotropic subspaces.

- A set S of totally isotropic d-spaces is an *EKR set* if no two elements intersect trivially.

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- *Polar disjointness graph*: vertices are totally isotropic d-spaces, adjacent when intersection is trivial.
- Hence EKR sets are cocliques in polar disjointness graph!
- Good candidate for largest construction: all totally isotropic d-spaces through fixed 1-space.

Applying Hoffman's bound to polar disjointness graph

Consider classical finite polar space with Witt index d, constructed in $V(n, q)$

- Eigenvalues of polar disjointness graph:

$$(-1)^j q^{d(d-1)/2+(d-j)(e-j)},\ j \in \{0, \ldots, d\},$$

where $e \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ depends on type polar space.

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  where $e \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ depends on type polar space.
- Max. eigenvalue for $j = 0$,
  min. eigenvalue usually for $j = 1$.
- Hoffman bound for EKR set S:

$$|S| \leq \frac{|\Omega|}{1 - k/\lambda_{min}},$$

  and χ_S is linear combination of eigenvectors in case of equality.
- Bound =
  usually number of totally isotropic d-spaces through 1-space.

Main result (Pepe, Storme, V. (2010))

Classification of EKR sets of maximum size
(i.e. sets of pairwise non-trivially intersecting d-spaces)
in all classical finite polar spaces of rank d,
except in $H(2d-1,q^2)/{}^2A_{2d-1}(q)$ for odd $d \geq 5$.

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Open case $H(2d-1,q^2)/{}^2A_{2d-1}(q)$ for odd $d \geq 5$

- Hoffman bound ($\sim q^{d^2-d}$) can never be attained.
- All through one 1-space ($\sim q^{(d-1)^2}$) seems best construction.
- Delsarte's linear programming bound (1973) can be stronger; for instance for $d = 5$:
  $(q+1)(q^7+1)(q^9+1) \sim q^{17}$,
  Hoffman: $\sim q^{20}$; through 1-space: $\sim q^{16}$.

Designs in general

- Consider a regular graph Γ with vertex set Ω.
- Let $V_0, \ldots, V_d$ denote *cometric ordering* of eigenspaces of Γ ("a meaningful ordering"), with $V_0 = \langle \chi_\Omega \rangle$.
- A subset $S \subseteq \Omega$ is a *t-design* if $\chi_S \in (V_1 \perp \ldots \perp V_t)^\perp$.

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- A subset $S \subseteq \Omega$ is a *t-design* if $\chi_S \in (V_1 \perp \ldots \perp V_t)^\perp$.
- Note: a t-design is also a t'-design if $t' \leq t$.
- t-designs usually have nice combinatorial characterizations using for instance Delsarte's *regular semilattices* (1975)

Example: designs in Hamming graph

- Hamming graph $H(d, q)$:
  vertices are codewords of length d over alphabet of size q,
  adjacent when differing in only one position.
- Let $V_0, \ldots, V_d$ denote eigenspaces of Γ,
  with descending order of eigenvalues.
- Here: S is t-design iff every incomplete word with t positions used appears in same number of words of S
  (i.e. S is an *orthogonal array* of strength t)
- For linear codes: t-designs are precisely
  duals of linear codes with minimal weight $t + 1$.

Example: designs in Johnson graph

- Johnson graph $J(n, d)$:
  vertices are subsets of size d in set X of size n,
  adjacent when intersection has size $d - 1$.
- Let $V_0, \ldots, V_d$ denote eigenspaces of Γ,
  with descending order of eigenvalues.
- Here: S is t-design iff every subset in X of size t
  is in same number of elements of S
  (i.e. S is a t-design in the classical sense)

Designs in Grassmann graph

- Grassmann graph $J_q(n, d)$:
  vertices are subspaces of dimension d in $V(n, q)$,
  adjacent when intersection has dimension $d - 1$.
- Let $V_0, \ldots, V_d$ denote eigenspaces of Γ,
  with descending order of eigenvalues.
- Here: S is t-design iff every subspace in $V(n, q)$ of dimension t
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  with descending order of eigenvalues.
- Here: S is t-design iff every subspace in $V(n, q)$ of dimension t
  is in same number of elements of S.
- Non-trivial designs: 1-designs quite easy,
  2-designs by for instance Thomas (1987), Suzuki (1990,1992),
  a 3-design by Braun-Kerber-Laue (2005)

Designs in Grassmann graph: properties

- Delsarte (1977): *design-orthogonal* subsets intersect nicely
- W.r.t. any non-degenerate alternating form on $V(2n, q)$: subset W of totally isotropic k-spaces satisfies: $\chi_W \in V_0 \perp V_2 \perp V_4 \perp \ldots$.
- If S is t-design of $(t+1)$-spaces in $V(2n, q)$, t even, $0 \leq t \leq n-1$, such that each t-space is in exactly λ elements of S:

$$|S \cap W| = \lambda\Big(\frac{q-1}{q^{2n-t}-1}\Big)\prod_{i=0}^{t}\frac{q^{2(n-i)}-1}{q^{i+1}-1}.$$

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$$|S \cap W| = \lambda\left(\frac{q-1}{q^{2n-t}-1}\right)\prod_{i=0}^{t}\frac{q^{2(n-i)}-1}{q^{i+1}-1}.$$

- Example: designs from Braun-Kerber-Laue (2005) with $n = 3, t = 2, q = 2, \lambda = 3$: $|S \cap W| = 27$.

Designs in dual polar graphs

Consider vector space
+ non-degen. quadratic/alternating/Hermitian form.

- Dual polar graph:
  vertices are totally isotropic subspaces of maximal dimension d,
  adjacent when intersecting in $(d-1)$-space
- Let $V_0, \ldots, V_d$ denote eigenspaces of Γ,
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  with descending order of eigenvalues.
- Here: S is t-design iff every totally isotropic t-space
  is in same number λ of elements of S.
- Many 1-designs are known (for instance *spreads* where $\lambda = 1$).
  No non-trivial t-designs with $t \geq 2$ seem known
  in non-bipartite dual polar graphs!

Designs in $H(2d-1,q^2)/{}^2A_{2d-1}(q)$

Here: consider $V(2d,q^2)$ with non-degenerate Hermitian form, and its totally isotropic d-spaces.

- Segre (1965):
  if $d=2$ and q odd, then a 1-design S of totally isotropic 2-spaces must be half of set of $(q+1)(q^3+1)$ vertices, known as *hemisystem*.
  Moreover, S induces a strongly regular graph

$$\mathrm{srg}\left(\frac{(q+1)(q^3+1)}{2},\frac{(q-1)(q^2+1)}{2},\frac{q-3}{2},\frac{(q-1)^2}{2}\right).$$

- Generalizations/alternative proofs: Bruen-Hirschfeld (1979), Cameron (1979), Thas (1981), Bamberg-Giudici-Royle (2010).

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- Segre (1965):
  unique hemisystem for $d=2, q=3$, inducing Gewirtz graph.
- Cossidente-Penttila (2005): hemisystems for all odd q ($d=2$).

$(d-1)$-designs in $H(2d-1,q^2)/{}^2A_{2d-1}(q)$, q odd

Here: consider $V(2d,q^2)$, q odd, with non-degenerate Hermitian form, and its totally isotropic d-spaces.

Generalization (V., 2011):
If S is half of set of totally isotropic d-spaces,
with each totally isotropic $(d-1)$-space in $\frac{q+1}{2}$ elements of S,
then S induces a *distance-regular subgraph with classical parameters* :

$$(d,-q,-(q+1)/2,-((-q)^d+1)/2).$$

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- No such graph known yet for $d \geq 3$.
- Existence of such design
  implies it for $H(2d'-1,q^2)/{}^2A_{2d'-1}(q)$ with $d' \leq d$.

Thank you for your attention!

Slides (and more) on `http://cage.ugent.be/~fvanhove`