

Eigenvalue techniques for regular and extremal substructures in geometry

Frédéric Vanhove

UGent (Belgium)

fvanhove@cage.ugent.be, <http://cage.ugent.be/~fvanhove>

July 11, 2011, Ghent

Outline

- Introduction: geometries, graphs
- Extremal (co)cliques: EKR
- Regular substructures: t -designs, hemisystems,....

Main idea

- 1 Consider a finite geometry (related to finite fields).
- 2 Associate graph(s) (with nice properties) with that geometry.
- 3 Use eigenvalues/eigenvectors of those graphs to consider substructures in geometry.

Example I: projective space from $V(n, q)$

- Objects are i -dimensional subspaces, with $i \in \{1, \dots, n\}$.
- Incidence I is just symmetrized inclusion between subspaces.

Example II: classical finite polar spaces

Consider $V(n, q)$ with non-degenerate quadratic/alternating/Hermitian form.

- Objects are i -dimensional subspaces, totally isotropic with respect to that form.
- Incidence I is just symmetrized inclusion between subspaces.
- The *rank* is the Witt index d : maximal dimension of totally isotropic subspaces.

Other examples

Axiomatically defined geometries, often constructed using finite fields: projective planes, generalized quadrangles/hexagons/octagons,...

Graphs

A *graph* Γ consists of:

- a finite set $\Omega \neq \emptyset$: the *vertices*
- a set E of pairs of vertices: the *edges*.

$x, y \in \Omega$ are *adjacent* or *neighbours* if $\{x, y\}$ is an edge of Γ .

We say Γ is *regular with valency k*
if every vertex has exactly k neighbours.

Eigenvalues and eigenvectors of graphs

Consider graph Γ with vertex set Ω :

- Adjacency matrix A : $(|\Omega| \times |\Omega|)$ -matrix over \mathbb{R} :

$$(A)_{x,y} = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are adjacent} \\ 0 & \text{if } x \text{ and } y \text{ are not adjacent} \end{cases} .$$

- *Eigenvalues of Γ* = the eigenvalues of A .
- A is real-symmetric \implies all eigenvalues are real.
- If Γ is regular with valency k then $A(1, 1, \dots, 1)^T = k(1, 1, \dots, 1)^T$, so k is eigenvalue (and largest eigenvalue).
- For any subset $S \subseteq \Omega$: *characteristic vector* $\chi_S = (1, 1, \dots, 0, 1)^T$ with 1 for elements of S , 0 for rest.

Cliques and cocliques

Consider any graph Γ with vertex set Ω .

- *Clique*: subset of vertices S , any two adjacent
- *Coclique*: subset of vertices S , no two adjacent

Question : how large can (co)cliques S in Γ be, and what if they are precisely that large?

General theorem on cocliques (Hoffman)

Suppose Γ is regular with valency $k \geq 1$, vertex set Ω and adjacency matrix A .

Suppose $S \subseteq \Omega$ is a coclique.

- If λ_{min} is minimal eigenvalue of Γ then

$$|S| \leq \frac{|\Omega|}{1 - k/\lambda_{min}}.$$

- In case of equality:
 χ_S is linear combination of eigenvectors of k and λ_{min} .

Original Erdős-Ko-Rado problem

- Erdős-Ko-Rado (1961): in a set X of size n , let S be a collection of subsets of size d ($n \geq 2d + 1$). If no two subsets in S are disjoint, then $|S| \leq \binom{n-1}{d-1}$, with equality if and only if all elements of S contain the same element of X .
- *Kneser graph* $K(n, d)$: vertices are subsets in X of size d , adjacent when intersection is empty.
- Hence we are studying cliques in $K(n, d)$!

Erdős-Ko-Rado problem for vector spaces over finite fields

- Hsieh (1975): in a vector space $V(n, q)$, let S be a collection of subspaces of dimension d ($n \geq 2d + 1$). If no two subspaces in S intersect trivially, then
$$|S| \leq \binom{n-1}{d-1}_q = \prod_{j=1}^{d-1} \frac{q^{n-j}-1}{q^j-1},$$
with equality iff all elements of S contain same 1-space in $V(n, q)$.
- q -Kneser graph $K_q(n, d)$: vertices are subspaces in $V(n, q)$ of dimension d , adjacent when intersection is trivial.
- Hence we are studying cocliques in $K_q(n, d)$!
- Improvements/alternative proofs: Green-Kleitman (1978), Frankl-Wilson (1986), Godsil-Newman (2006), Tanaka (2006)

Erdős-Ko-Rado problem for classical finite polar spaces

Consider finite-dimensional vector space

+ non-degen. quadratic/alternating/Hermitian form with Witt index d .

Polar space consists of totally isotropic subspaces.

- A set S of totally isotropic d -spaces is an *EKR set* if no two elements intersect trivially.
- *Polar disjointness graph*: vertices are totally isotropic d -spaces, adjacent when intersection is trivial.
- Hence EKR sets are cliques in polar disjointness graph!
- Good candidate for largest construction:
all totally isotropic d -spaces through fixed 1-space.

Applying Hoffman's bound to polar disjointness graph

Consider classical finite polar space with Witt index d , constructed in $V(n, q)$

- Eigenvalues of polar disjointness graph:

$$(-1)^j q^{d(d-1)/2+(d-j)(e-j)}, \quad j \in \{0, \dots, d\},$$

where $e \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ depends on type polar space.

- Max. eigenvalue for $j = 0$,
min. eigenvalue usually for $j = 1$.
- Hoffman bound for EKR set S :

$$|S| \leq \frac{|\Omega|}{1 - k/\lambda_{min}},$$

and χ_S is linear combination of eigenvectors in case of equality.

- Bound =
usually number of totally isotropic d -spaces through 1-space.

Main result (Pepe, Storme, V. (2010))

Classification of EKR sets of maximum size
(i.e. sets of pairwise non-trivially intersecting d -spaces)
in all classical finite polar spaces of rank d ,
except in $H(2d - 1, q^2)/{}^2A_{2d-1}(q)$ for odd $d \geq 5$.

Open case $H(2d - 1, q^2)/{}^2A_{2d-1}(q)$ for odd $d \geq 5$

- Hoffman bound ($\sim q^{d^2-d}$) can never be attained.
- All through one 1-space ($\sim q^{(d-1)^2}$) seems best construction.
- Delsarte's linear programming bound (1973) can be stronger;
for instance for $d = 5$:
 $(q + 1)(q^7 + 1)(q^9 + 1) \sim q^{17}$,
Hoffman: $\sim q^{20}$; through 1-space: $\sim q^{16}$.

Designs in general

- Consider a regular graph Γ with vertex set Ω .
- Let V_0, \dots, V_d denote *cometric ordering* of eigenspaces of Γ (“a meaningful ordering”), with $V_0 = \langle \chi_\Omega \rangle$.
- A subset $S \subseteq \Omega$ is a *t-design* if $\chi_S \in (V_1 \perp \dots \perp V_t)^\perp$.
- Note: a *t*-design is also a *t'*-design if $t' \leq t$.
- *t*-designs usually have nice combinatorial characterizations using for instance Delsarte’s *regular semilattices* (1975)

Example: designs in Hamming graph

- Hamming graph $H(d, q)$:
vertices are codewords of length d over alphabet of size q ,
adjacent when differing in only one position.
- Let V_0, \dots, V_d denote eigenspaces of Γ ,
with descending order of eigenvalues.
- Here: S is t -design iff every incomplete word with t positions used
appears in same number of words of S
(i.e. S is an *orthogonal array* of strength t)
- For linear codes: t -designs are precisely
duals of linear codes with minimal weight $t + 1$.

Example: designs in Johnson graph

- Johnson graph $J(n, d)$:
vertices are subsets of size d in set X of size n ,
adjacent when intersection has size $d - 1$.
- Let V_0, \dots, V_d denote eigenspaces of Γ ,
with descending order of eigenvalues.
- Here: S is t -design iff every subset in X of size t
is in same number of elements of S
(i.e. S is a t -design in the classical sense)

Designs in Grassmann graph

- Grassmann graph $J_q(n, d)$:
vertices are subspaces of dimension d in $V(n, q)$,
adjacent when intersection has dimension $d - 1$.
- Let V_0, \dots, V_d denote eigenspaces of Γ ,
with descending order of eigenvalues.
- Here: S is t -design iff every subspace in $V(n, q)$ of dimension t
is in same number of elements of S .
- Non-trivial designs: 1-designs quite easy,
2-designs by for instance Thomas (1987), Suzuki (1990,1992),
a 3-design by Braun-Kerber-Laue (2005)

Designs in Grassmann graph: properties

- Delsarte (1977): *design-orthogonal* subsets intersect nicely
- W.r.t. any non-degenerate alternating form on $V(2n, q)$: subset W of totally isotropic k -spaces satisfies:
 $\chi_W \in V_0 \perp V_2 \perp V_4 \perp \dots$
- If S is t -design of $(t+1)$ -spaces in $V(2n, q)$, t even, $0 \leq t \leq n-1$, such that each t -space is in exactly λ elements of S :

$$|S \cap W| = \lambda \left(\frac{q-1}{q^{2n-t}-1} \right) \prod_{i=0}^t \frac{q^{2(n-i)}-1}{q^{i+1}-1}.$$

- Example: designs from Braun-Kerber-Laue (2005) with $n=3, t=2, q=2, \lambda=3$: $|S \cap W| = 27$.

Designs in dual polar graphs

Consider vector space

+ non-degen. quadratic/alternating/Hermitian form.

- Dual polar graph:
vertices are totally isotropic subspaces of maximal dimension d ,
adjacent when intersecting in $(d - 1)$ -space
- Let V_0, \dots, V_d denote eigenspaces of Γ ,
with descending order of eigenvalues.
- Here: S is t -design iff every totally isotropic t -space
is in same number λ of elements of S .
- Many 1-designs are known (for instance *spreads* where $\lambda = 1$).
No non-trivial t -designs with $t \geq 2$ seem known
in non-bipartite dual polar graphs!

Designs in $H(2d - 1, q^2)/{}^2A_{2d-1}(q)$

Here: consider $V(2d, q^2)$ with non-degenerate Hermitian form, and its totally isotropic d -spaces.

- Segre (1965):

if $d = 2$ and q odd, then a 1-design S of totally isotropic 2-spaces must be half of set of $(q + 1)(q^3 + 1)$ vertices, known as *hemisystem*. Moreover, S induces a strongly regular graph

$$\text{srg}\left(\frac{(q + 1)(q^3 + 1)}{2}, \frac{(q - 1)(q^2 + 1)}{2}, \frac{q - 3}{2}, \frac{(q - 1)^2}{2}\right).$$

- Generalizations/alternative proofs: Bruen-Hirschfeld (1979), Cameron (1979), Thas (1981), Bamberg-Giudici-Royle (2010).
- Segre (1965):
unique hemisystem for $d = 2, q = 3$, inducing Gewirtz graph.
- Cossidente-Penttila (2005): hemisystems for all odd q ($d = 2$).

$(d - 1)$ -designs in $H(2d - 1, q^2)/{}^2A_{2d-1}(q)$, q odd

Here: consider $V(2d, q^2)$, q odd, with non-degenerate Hermitian form, and its totally isotropic d -spaces.

Generalization (V., 2011):

If S is half of set of totally isotropic d -spaces, with each totally isotropic $(d - 1)$ -space in $\frac{q+1}{2}$ elements of S , then S induces a *distance-regular subgraph with classical parameters* :

$$(d, -q, -(q + 1)/2, -((-q)^d + 1)/2).$$

- No such graph known yet for $d \geq 3$.
- Existence of such design implies it for $H(2d' - 1, q^2)/{}^2A_{2d'-1}(q)$ with $d' \leq d$.

Thank you for your attention!

Slides (and more) on <http://cage.ugent.be/~fvanhove>