Extremal and regular subsets in regular near polygons

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Outline

- Introduction: (regular) near $2d$-gons, association schemes, distance-regular graphs....
- Substructures in regular near $2d$-gons
- Generalized polygons
- Dual polar spaces

Point-line geometry is ordered triple (P, L, I), $P, L \neq \emptyset, \mathrm{I} \subseteq (P \times L)$.

- If $(p, \ell) \in \mathrm{I}$ then p *is on* ℓ, or ℓ *contains* p.
- *Collinearity graph* has vertex set P, with 2 points adjacent when on common line.

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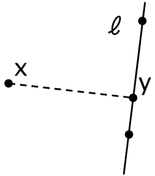
Definition

Near $2d$-gons, $d \geq 2$, are point-line geometries:

1. 2 points are on at most 1 line (every line contains at least 2 points),
2. the collinearity graph on points has diameter d,
3. $\forall$ point x and $\forall$ line ℓ, there is unique point $y \in \ell$ at min. distance from x.



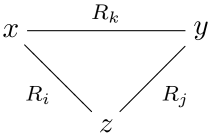
Simplest example: ordinary $2d$-gon!

We will want high regularity!

Association schemes

$(\Omega, \{R_0, \ldots, R_d\})$, with $\Omega \neq \emptyset$ finite set, is association scheme if:

- $\{R_0, \ldots, R_d\}$ partitions $\Omega \times \Omega$,
- R_0 is identity relation,
- $(\omega_1, \omega_2) \in R_i \Longleftrightarrow (\omega_2, \omega_1) \in R_i$,
- there are *intersection numbers* p_{ij}^k:
  if $(x, y) \in R_k$, the number of elements z in Ω
  for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is p_{ij}^k.



Definition of matrices A_i

Consider association scheme $(\Omega, \{R_0, \dots, R_d\})$
and order the elements of Ω: $\omega_1, \dots, \omega_{|\Omega|}$.
For each R_i, define real $(|\Omega| \times |\Omega|)$-matrix A_i:

$$\begin{cases} (A_i)_{rs} = 1 \text{ if } (\omega_r, \omega_s) \in R_i, \\ (A_i)_{rs} = 0 \text{ if } (\omega_r, \omega_s) \notin R_i. \end{cases}$$

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Properties

- $A_0 + \ldots + A_d$ is all-one matrix.
- A_0 is identity matrix.
- A_i is symmetric.
- $A_i A_j = \sum_k p_{ij}^k A_k$.

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Bose-Mesner algebra: algebra with basis $\{A_0, \dots, A_d\}$.

Eigenvectors for scheme $(\Omega, \{R_0, \dots, R_d\})$

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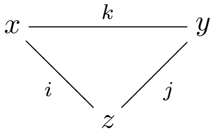
Idempotents

- Orthogonal projection E_j onto V_j is also in $\langle A_0, \ldots, A_d \rangle$.
- These *minimal idempotents* form second basis $\{E_0, \ldots, E_d\}$ (we let E_0 denote projection onto all-one vector).

Distance-regular graphs

- Consider graph $\Gamma = (\Omega, E)$ of diameter d.
- Write $\Gamma_i(x)$ for vertices at distance i from x.
- Define R_i as $\{(x, y) \in \Omega \times \Omega | d(x, y) = i\}$.
- Γ is *distance-regular* if $(\Omega, \{R_0, \dots, R_d\})$ is association scheme.



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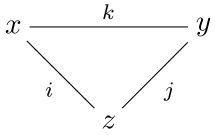

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Convention:

$$c_k = |\Gamma_1(x) \cap \Gamma_{k-1}(y)|, a_k = |\Gamma_1(x) \cap \Gamma_k(y)|, b_k = |\Gamma_1(x) \cap \Gamma_{k+1}(y)|.$$

Such association schemes are called *P-polynomial* or *metric*.

Q-polynomial association schemes

- Bose-Mesner algebra $\langle A_0, \ldots, A_d \rangle = \langle E_0, \ldots, E_d \rangle$ is also closed under entrywise multiplication "$\circ$".
- $E_0, \ldots, E_d$ is a *Q-polynomial* or *cometric* ordering if:

$$|\Omega|(E_1 \circ E_i) = b^*_{i-1} E_{i-1} + a^*_i E_i + c^*_{i+1} E_{i+1},$$

with $c^*_i \neq 0$ for $i \in \{1, \ldots, d\}$.

Recall: Near $2d$-gons, $d \geq 2$, are point-line geometries:

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Near $2d$-gon is *regular* if the collinearity graph is distance-regular.
$\Longrightarrow$ Then it has an *order* $(s, t), s, t \geq 1$:
$s + 1$ points on each line, $t + 1$ lines through each point.

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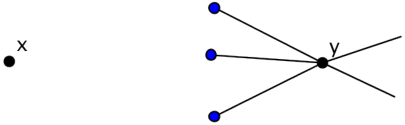
If $d(x,y) = i$ then through y:

- c_i lines with 1 point at distance $i-1$ from x, and its s other points at distance i from x,
- $(t+1) - c_i$ lines at distance i from x.



Some types of regular near $2d$-gons of order (s, t):

Generalized $2d$-gons

For every point p and line ℓ, there is a unique shortest path from p to ℓ.
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Dual polar spaces

Consider vector space with non-degenerate quadratic/alternating/Hermitian form of Witt index d.
Dual polar space:

- "points": totally isotropic d-spaces
- "line": totally isotropic $(d-1)$-spaces

Here $c_i = (q^i - 1)/(q - 1)$ and $s = q^e$.

Consider association scheme $(\Omega, \{R_0, \ldots, R_d\})$ and non-empty $S \subseteq \Omega$.

Counting with respect to S

- inner distribution $\mathbf{a}$: $\mathbf{a}_i = \frac{|R_i \cap (S \times S)|}{|S|}$ ("average inner valency")
- outer distribution B: $B_{\omega,i} = |R_i \cap (\{\omega\} \times S)|$
- characteristic vector of subset $S \subseteq \Omega$:

$$\chi_S = (1, 1, 0, \ldots, 1, 0)^T \in \mathbb{R}^\Omega,$$

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Consider minimal idempotent $E = (\theta_0 A_0 + \ldots + \theta_d A_d)/|\Omega|$

Delsarte (1973)

$$\theta_0 \mathbf{a}_0 + \ldots + \theta_d \mathbf{a}_d \geq 0,$$

with equality iff $E\chi_S = 0$ (i.e. iff $\chi_S \in \mathrm{Im}(E)^\perp$), and then:

$$\theta_0 B_{\omega,0} + \ldots + \theta_d B_{\omega,d} = 0, \forall \omega \in \Omega.$$

Consider association scheme $(\Omega, \{R_0, \ldots, R_d\})$ and $S_1, S_2 \subseteq \Omega$.
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Delsarte (1977): Design-orthogonality

Observe: $|S_1 \cap S_2| = (\chi_{S_1})^T \chi_{S_2}$.

S_1 and S_2 are *design-orthogonal* if $\forall j \neq 0$:$E_j \chi_{S_1} = 0$ or $E_j \chi_{S_2} = 0$.
Then: $|S_1 \cap S_2| = |S_1||S_2|/|\Omega|$.

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Martin (2001): Vanishing Krein parameters q_{ij}^k

Observe $\chi_{S_1 \cap S_2} = \chi_{S_1} \circ \chi_{S_2}$ (entrywise product).

If $(E_i \circ E_j)E_k = 0$ (i.e. if $q_{ij}^k = 0$) then

$$E_k(u \circ v) = 0, \forall u \in \mathrm{Im}(E_i), \forall v \in \mathrm{Im}(E_j).$$

$\Longrightarrow$ properties of $\chi_{S_1 \cap S_2}$

Consider collinearity graph Γ of regular near $2d$-gon of order (s, t).

Eigenvalues of Γ

- Largest eigenvalue: valency $s(t+1)$.
  Eigenspace: spanned by all-one vector χ_Ω
- Smallest eigenvalue: $-(t+1)$.
  Eigenspace: kernel incidence matrix between points and lines
  Corresponding idempotent is (up to positive scalar):

$$\frac{A_0}{1} + \frac{A_1}{-s} + \ldots + \frac{A_d}{(-s)^d}.$$

Consider a regular near $2d$-gon of order (s, t).

Tight subsets of points

Let E be idempotent corresponding to eigenvalue $-(t+1)$ of collinearity graph.

A set of points S is *tight* if $E\chi_S = 0$.

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Properties of tight sets

- For every point p: $\sum_{i=0}^{d} B_{p,i}/(-s)^i = 0$
  ($B_{p,i}$ is number of points in S at distance i from p)
- $|S| = i(s+1)$ for some integer i (we say S is i-tight).
- Any union of m disjoint lines is m-tight.
- 1-tight set = set of $s+1$ points on some line.

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Payne (1987): introduced tight sets for regular near 4-gons.

Consider a regular near $2d$-gon of order (s, t).
Let E_0 be trivial idempotent, E be idempotent for eigenvalue $-(t+1)$.

m-ovoids

S is an *m-ovoid* if every line intersects S in exactly m points.
Since $\mathrm{Im}(E)$ is kernel of incidence matrix:

$$S \text{ is } m\text{-ovoid} \iff \chi_S = E_0\chi_S + E\chi_S.$$

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Tight set versus m-ovoid

An i-tight set S_1 is design-orthogonal to m-ovoid S_2 with $|S_1 \cap S_2| = mi$.

Similar concepts/results for partial geometries, projective and polar spaces by Eisfeld (1998), Bamberg-Kelly-Law-Penttila (2007), De Wispelaere-Van Maldeghem(2008), Bamberg-Law-Penttila (2009)

Consider generalized 6-gon of order (s, t) (so $c_2 = 1$).
If $t = s^3, s > 1 \Longrightarrow s(s^3+1), -s^3-1, s^2+s-1, -s^2+s-1$ is Q-polynomial ordering eigenvalues of collinearity graph

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Intersecting 1-ovoids S_1, S_2 for order (s, s^3)

- 1-ovoid: coclique s.t. every line contains 1 point
- Two 1-ovoids S_1, S_2 satisfy: χ_{S_1}, χ_{S_2} in span of first 2 eigenspaces
- $\chi_{S_1 \cap S_2}$ orthogonal to last eigenspace, so every $p \in S_1 \cap S_2$ satisfies:

$$B_{p,.} = \left(1, 0, \frac{(s+1)|S_1 \cap S_2|}{s^2+s+1} + (s^2-1)(s^3+1), s^2\left(\frac{|S_1 \cap S_2|}{s^2+s+1} - (s^3-s+1)\right)\right).$$

- Hence $|S_1 \cap S_2| = 0$ or $h(s^2+s+1)$ for some integer $h \geq s^3 - s + 1$.

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Examples?

Only known generalized hexagons of order $(s,s^3), s>1$, are *dual twisted-triality hexagons* $T(q,q^3)$ with q prime power.
De Wispelaere-Van Maldeghem (2004): no 1-ovoids in $\mathrm{T}(2,8)$, $\mathrm{T}(3,27)$.

Consider generalized octagon of order (s, t), $s > 1$ (so $c_2 = c_3 = 1$).

Suboctagons

- Let S be point-set of suboctagon of order (s', t').
- Inner distribution $\mathbf{a}$ consists of valencies suboctagon.
- Use idempotent for eigenvalue $-(t+1)$:

$$\sum_{i=0}^{4} \mathbf{a}_i/(-s)^i = \frac{(s-s')(s-s't')(s^2+(s't')^2)}{s^4} \geq 0,$$

  with equality iff S is tight.
- Thas (1979): $(s = s') \Longrightarrow t' = 1$, so $s \geq s't'$ holds in any case.

Consider vector space with non-degenerate form with Witt index d.
Points dual polar space = totally isotropic d-spaces

Eigenspaces

- Q-polynomial ordering of idempotents $E_0, \ldots, E_d$ ("regular semilattice ordering"), E_d corresponds to $\lambda_d = -(t+1)$.
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The case $\chi_S = E_0\chi_S + E_1\chi_S + E_d\chi_S$

- *dual zero intervals*, see Suda (2011)
- S is "nicer" with respect to $(d-1)$-spaces than to d-spaces
- Similar ideas by Calderbank-Delsarte (1993) and Delsarte (2004)
- Application: Pepe-Storme-V. (2011): extremal Erdős-Ko-Rado subsets in $C_d(q)/W(2d-1,q)$ for odd d. (descendents in Ustimenko graph, see Tanaka (201?))

Polar space from Hermitian form on $V(2d, q^2)$: ${}^2A_{2d-1}(q)/H(2d-1, q^2)$.
Consider subset S of totally isotropic d-spaces.

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- Here: S is m-ovoid iff $\chi_S = E_0\chi_S + E_d\chi_S$ (i. e. every $(d-1)$-space in exactly m elements of S).
- Segre (1965): non-trivial S must be half of set of d-spaces: $m = (q+1)/2$
- V. (2011): if S is such a half, it induces distance-regular graph with classical parameters $(d, -q, -(q+1)/2, -((-q)^d + 1)/2)$.
- Cossidente-Penttila (2005): construction of such “hemisystems” for odd q and $d = 2$.
- Open problem for $d \geq 3$!

Special property of ${}^2A_{2d-1}(q)/H(2d-1,q^2)$:
two Q-polynomial orderings!

The case $d = 3$

- orderings: E_0, E_1, E_2, E_3(semilattices) and E_0, E_3, E_1, E_2.

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- If S_1 and S_2 are $(q+1)/2$-ovoids, then 2nd ordering implies:

$$\begin{aligned}\chi_{S_1\cap S_2} &= E_0\chi_{S_1\cap S_2} + E_3\chi_{S_1\cap S_2} + E_1\chi_{S_1\cap S_2} \\ &= (E_0\chi_{S_1\cap S_2} + E_1\chi_{S_1\cap S_2}) + E_3\chi_{S_1\cap S_2},\end{aligned}$$

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- $\Longrightarrow$ if line ℓ of dual polar space contains x points of $S_1 \cap S_2$:

$$x(q^3-1) + \frac{|S_1 \cap S_2|}{q^3+1},$$

elements of $S_1 \cap S_2$ at distance 1 from ℓ in the near hexagon.
$\Longrightarrow |S_1 \cap S_2|$ is divisible by $q^3 + 1$.

Thank you for your attention!

Slides (and more) on `http://cage.ugent.be/~fvanhove`