Extremal and regular subsets in regular near polygons

Frédéric Vanhove
UGent (Belgium)
fvanhove@cage.ugent.be, http://cage.ugent.be/~fvanhove

August 16, 2011, GAC5
Outline

- Introduction: (regular) near $2d$-gons, association schemes, distance-regular graphs....
- Substructures in regular near $2d$-gons
- Generalized polygons
- Dual polar spaces
**Introduction**

Substructures in general near 2d-gons

Generalized 2d-gons

Dual polar spaces

Near 2d-gons

---

**Point-line geometry** is ordered triple \((P, L, I)\), \(P, L \neq \emptyset\), \(I \subseteq (P \times L)\).

- If \((p, \ell) \in I\) then \(p\) is on \(\ell\), or \(\ell\) contains \(p\).
- **Collinearity graph** has vertex set \(P\), with 2 points adjacent when on common line.
**Point-line geometry** is ordered triple \((P, L, I)\), \(P, L \neq \emptyset, I \subseteq (P \times L)\).

- If \((p, \ell) \in I\) then \(p\) is on \(\ell\), or \(\ell\) contains \(p\).
- **Collinearity graph** has vertex set \(P\),
  with 2 points adjacent when on common line.

**Definition**

Near \(2d\)-gons, \(d \geq 2\), are point-line geometries:

1. 2 points are on at most 1 line
   (every line contains at least 2 points),
2. the collinearity graph on points has diameter \(d\),
3. \(\forall\) point \(x\) and \(\forall\) line \(\ell\),
   there is unique point \(y \in \ell\)
   at min. distance from \(x\).

Simplest example: ordinary \(2d\)-gon!

We will want high regularity!

Frédéric Vanhove (UGent)
Association schemes

$(\Omega, \{R_0, \ldots, R_d\})$, with $\Omega \neq \emptyset$ finite set, is association scheme if:

- $\{R_0, \ldots, R_d\}$ partitions $\Omega \times \Omega$,
- $R_0$ is identity relation,
- $(\omega_1, \omega_2) \in R_i \iff (\omega_2, \omega_1) \in R_i$,
- there are intersection numbers $p_{ij}^k$: if $(x, y) \in R_k$, the number of elements $z$ in $\Omega$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ is $p_{ij}^k$.

\[ x \xleftarrow{R_k} y \]
\[ \Downarrow \quad \Downarrow \quad \Downarrow \]
\[ \quad \quad \quad \quad z \]
\[ R_i \quad \quad R_j \]
**Definition of matrices $A_i$**

Consider association scheme $(\Omega, \{R_0, \ldots, R_d\})$ and order the elements of $\Omega$: $\omega_1, \ldots, \omega_{|\Omega|}$.

For each $R_i$, define real $(|\Omega| \times |\Omega|)$-matrix $A_i$:

$$
\begin{align*}
(A_i)_{rs} &= 1 \text{ if } (\omega_r, \omega_s) \in R_i, \\
(A_i)_{rs} &= 0 \text{ if } (\omega_r, \omega_s) \notin R_i.
\end{align*}
$$
**Definition of matrices $A_i$**

Consider association scheme $(\Omega, \{R_0, \ldots, R_d\})$ and order the elements of $\Omega$: $\omega_1, \ldots, \omega_{|\Omega|}$.

For each $R_i$, define real $(|\Omega| \times |\Omega|)$-matrix $A_i$:

\[
\begin{cases} 
(A_i)_{rs} = 1 & \text{if } (\omega_r, \omega_s) \in R_i, \\
(A_i)_{rs} = 0 & \text{if } (\omega_r, \omega_s) \not\in R_i.
\end{cases}
\]

**Properties**

- $A_0 + \ldots + A_d$ is all-one matrix.
- $A_0$ is identity matrix.
- $A_i$ is symmetric.
- $A_i A_j = \sum_k p_{ij}^k A_k$. 

---

Frédéric Vanhove (UGent) Subsets in regular near 2d-gons August 16, 2011, GAC5
Definition of matrices $A_i$

Consider association scheme $(\Omega, \{R_0, \ldots, R_d\})$
and order the elements of $\Omega$: $\omega_1, \ldots, \omega_{|\Omega|}$.
For each $R_i$, define real $(|\Omega| \times |\Omega|)$-matrix $A_i$:

$$
\begin{cases}
(A_i)_{rs} = 1 & \text{if } (\omega_r, \omega_s) \in R_i, \\
(A_i)_{rs} = 0 & \text{if } (\omega_r, \omega_s) \notin R_i.
\end{cases}
$$

Properties

- $A_0 + \ldots + A_d$ is all-one matrix.
- $A_0$ is identity matrix.
- $A_i$ is symmetric.
- $A_i A_j = \sum_k p_{ij}^k A_k$.

*Bose-Mesner algebra*: algebra with basis $\{A_0, \ldots, A_d\}$. 
Eigenvectors for scheme \((\Omega, \{R_0, \ldots, R_d\})\)

\(\mathbb{R}^\Omega\): real vector space with orthonormal basis indexed by elements of \(\Omega\).
Eigenvectors for scheme \((\Omega, \{R_0, \ldots, R_d\})\)

\(\mathbb{R}^\Omega\): real vector space with orthonormal basis indexed by elements of \(\Omega\).
\(\mathbb{R}^\Omega\) uniquely decomposes as:

\[
\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d,
\]

with every \(v \neq 0 \in V_j\) an eigenvector of every relation \(R_i\): \(A_i v = \lambda_{ji} v\).
Eigenvectors for scheme \((\Omega, \{R_0, \ldots, R_d\})\)

\(\mathbb{R}^\Omega\): real vector space with orthonormal basis indexed by elements of \(\Omega\).  
\(\mathbb{R}^\Omega\) uniquely decomposes as:

\[
\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d,
\]

with every \(v \neq 0 \in V_j\) an eigenvector of every relation \(R_i: A_i v = \lambda_{ji} v\).

Idempotents

- Orthogonal projection \(E_j\) onto \(V_j\) is also in \(\langle A_0, \ldots, A_d \rangle\).
- These minimal idempotents form second basis \(\{E_0, \ldots, E_d\}\) (we let \(E_0\) denote projection onto all-one vector).
Distance-regular graphs

- Consider graph $\Gamma = (\Omega, E)$ of diameter $d$.
- Write $\Gamma_i(x)$ for vertices at distance $i$ from $x$.
- Define $R_i$ as $\{(x, y) \in \Omega \times \Omega | d(x, y) = i\}$.
- $\Gamma$ is distance-regular if $(\Omega, \{R_0, \ldots, R_d\})$ is association scheme.

\[
\begin{array}{c}
  x \\
  \downarrow^i \\
  z \\
  \downarrow^j \\
  y\\
\end{array}
\]

So number of $z$ in $\Gamma_i(x) \cap \Gamma_j(y)$ must be constant $p_{ij}^k$. 
Distance-regular graphs

- Consider graph $\Gamma = (\Omega, E)$ of diameter $d$.
- Write $\Gamma_i(x)$ for vertices at distance $i$ from $x$.
- Define $R_i$ as $\{(x, y) \in \Omega \times \Omega | d(x, y) = i\}$.
- $\Gamma$ is distance-regular if $(\Omega, \{R_0, \ldots, R_d\})$ is association scheme.

So number of $z$ in $\Gamma_i(x) \cap \Gamma_j(y)$ must be constant $p_{ij}^k$.

Convention:

$$c_k = |\Gamma_1(x) \cap \Gamma_{k-1}(y)|, \quad a_k = |\Gamma_1(x) \cap \Gamma_k(y)|, \quad b_k = |\Gamma_1(x) \cap \Gamma_{k+1}(y)|.$$  

Such association schemes are called $P$-polynomial or metric.
Q-polynomial association schemes

- Bose-Mesner algebra $\langle A_0, \ldots, A_d \rangle = \langle E_0, \ldots, E_d \rangle$ is also closed under entrywise multiplication “$\circ$”.

- $E_0, \ldots, E_d$ is a Q-polynomial or cometric ordering if:

$$|\Omega|(E_1 \circ E_i) = b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1},$$

with $c_i^* \neq 0$ for $i \in \{1, \ldots, d\}$. 
Regular near $2d$-gons

Recall: Near $2d$-gons, $d \geq 2$, are point-line geometries:

1. 2 points on at most one line (every line contains at least 2 points),
2. the collinearity graph on points has diameter $d$,
3. $\forall$ point $x$ and $\forall$ line $\ell$, there is unique point $y \in \ell$ at minimal distance from $x$. 
Recall: Near $2d$-gons, $d \geq 2$, are point-line geometries:

1. 2 points on at most one line (every line contains at least 2 points),
2. the collinearity graph on points has diameter $d$,
3. $\forall$ point $x$ and $\forall$ line $\ell$, there is unique point $y \in \ell$ at minimal distance from $x$.

Near $2d$-gon is *regular* if the collinearity graph is distance-regular.

$\implies$ Then it has an *order* $(s, t)$, $s, t \geq 1$:

$s + 1$ points on each line, $t + 1$ lines through each point.
Recall: Near 2d-gons, \(d \geq 2\), are point-line geometries:

1. 2 points on at most one line (every line contains at least 2 points),
2. the collinearity graph on points has diameter \(d\),
3. \(\forall\) point \(x\) and \(\forall\) line \(\ell\), there is unique point \(y \in \ell\) at minimal distance from \(x\).

Near 2d-gon is regular if the collinearity graph is distance-regular.

\[\implies\] Then it has an order \((s, t)\), \(s, t \geq 1\):

\[s + 1\] points on each line, \(t + 1\) lines through each point.

If \(d(x, y) = i\) then through \(y\):

- \(c_i\) lines with 1 point at distance \(i - 1\) from \(x\),
- and its \(s\) other points at distance \(i\) from \(x\),
- \((t + 1) - c_i\) lines at distance \(i\) from \(x\).
Some types of regular near $2d$-gons of order $(s, t)$:

**Generalized $2d$-gons**

For every point $p$ and line $\ell$, there is a unique shortest path from $p$ to $\ell$. Here $c_1 = \ldots = c_{d-1} = 1$ and $c_d = t + 1$. 
Some types of regular near $2d$-gons of order $(s, t)$:

**Generalized $2d$-gons**

For every point $p$ and line $\ell$, there is a unique shortest path from $p$ to $\ell$. Here $c_1 = \ldots = c_{d-1} = 1$ and $c_d = t + 1$.

**Dual polar spaces**

Consider vector space with non-degenerate quadratic/alternating/Hermitian form of Witt index $d$. 

*Dual polar space:*

- “points”: totally isotropic $d$-spaces
- “line”: totally isotropic $(d - 1)$-spaces

Here $c_i = (q^i - 1)/(q - 1)$ and $s = q^e$. 

Frédéric Vanhove (UGent)
Consider association scheme \((\Omega, \{R_0, \ldots, R_d\})\) and non-empty \(S \subseteq \Omega\).

**Counting with respect to \(S\)**

- **inner distribution** \(\mathbf{a}\): \(a_i = \frac{|R_i \cap (S \times S')|}{|S|}\) ("average inner valency")
- **outer distribution** \(B\): \(B_{\omega,i} = |R_i \cap (\{\omega\} \times S)|\)
- **characteristic vector of subset** \(S \subseteq \Omega\):

\[
\chi_S = (1, 1, 0, \ldots, 1, 0)^T \in \mathbb{R}^\Omega,
\]
Consider association scheme \((\Omega, \{R_0, \ldots, R_d\})\) and non-empty \(S \subseteq \Omega\).

**Counting with respect to** \(S\)

- **inner distribution** \(\mathbf{a}: \mathbf{a}_i = \frac{|R_i \cap (S \times S)|}{|S|}\) ("average inner valency")
- **outer distribution** \(B: B_{\omega,i} = |R_i \cap (\{\omega\} \times S)|\)
- **characteristic vector of subset** \(S \subseteq \Omega:\)

\[
\chi_S = (1, 1, 0, \ldots, 1, 0)^T \in \mathbb{R}^\Omega,
\]

Consider minimal idempotent \(E = (\theta_0 A_0 + \ldots + \theta_d A_d)/|\Omega|\)

*Delsarte (1973)*

\[
\theta_0 \mathbf{a}_0 + \ldots + \theta_d \mathbf{a}_d \geq 0,
\]

with equality iff \(E \chi_S = 0\) (i.e. iff \(\chi_S \in \text{Im}(E)^\perp\)), and then:

\[
\theta_0 B_{\omega,0} + \ldots + \theta_d B_{\omega,d} = 0, \forall \omega \in \Omega.
\]
Consider association scheme \((\Omega, \{R_0, \ldots, R_d\})\) and \(S_1, S_2 \subseteq \Omega\).

What about \(S_1 \cap S_2\)?
Consider association scheme \((\Omega, \{R_0, \ldots, R_d\})\) and \(S_1, S_2 \subseteq \Omega\).

What about \(S_1 \cap S_2\)?

**Delsarte (1977): Design-orthogonality**

Observe: \(|S_1 \cap S_2| = (\chi_{S_1})^T \chi_{S_2}\).

\(S_1\) and \(S_2\) are *design-orthogonal* if \(\forall j \neq 0: E_j \chi_{S_1} = 0\) or \(E_j \chi_{S_2} = 0\).

Then: \(|S_1 \cap S_2| = |S_1||S_2|/|\Omega|\).
Consider association scheme \((\Omega, \{R_0, \ldots, R_d\})\) and \(S_1, S_2 \subseteq \Omega\). What about \(S_1 \cap S_2\)?

**Delsarte (1977): Design-orthogonality**

Observe: \(|S_1 \cap S_2| = (\chi_{S_1})^T \chi_{S_2}|.

\(S_1\) and \(S_2\) are *design-orthogonal* if \(\forall j \neq 0: E_j \chi_{S_1} = 0\) or \(E_j \chi_{S_2} = 0\).

Then: \(|S_1 \cap S_2| = |S_1||S_2|/|\Omega|\).

**Martin (2001): Vanishing Krein parameters \(q_{ij}^k\)**

Observe \(\chi_{S_1 \cap S_2} = \chi_{S_1} \circ \chi_{S_2}\) (entrywise product).

If \((E_i \circ E_j)E_k = 0\) (i.e. if \(q_{ij}^k = 0\) then

\[E_k(u \circ v) = 0, \forall u \in \text{Im}(E_i), \forall v \in \text{Im}(E_j).\]

\(\implies\) properties of \(\chi_{S_1 \cap S_2}\)
Consider collinearity graph $\Gamma$ of regular near $2d$-gon of order $(s, t)$.

**Eigenvalues of $\Gamma$**

- Largest eigenvalue: valency $s(t + 1)$.
  Eigenspace: spanned by all-one vector $\chi_\Omega$

- Smallest eigenvalue: $-(t + 1)$.
  Eigenspace: kernel incidence matrix between points and lines

Corresponding idempotent is (up to positive scalar):

$$\frac{A_0}{1} + \frac{A_1}{-s} + \ldots + \frac{A_d}{(-s)^d}.$$
Tight sets and $m$-ovoids in regular near $2d$-gons

Consider a regular near $2d$-gon of order $(s, t)$.

**Tight subsets of points**

Let $E$ be idempotent corresponding to eigenvalue $-(t + 1)$ of collinearity graph.

A set of points $S$ is *tight* if $E \chi_S = 0$.

- For $S \neq \emptyset$, we have $\sum_{i=0}^{d} a_i/(-s)^i \geq 0$, with equality iff $S$ is tight.
Consider a regular near $2d$-gon of order $(s, t)$.

Tight subsets of points

Let $E$ be idempotent corresponding to eigenvalue $-(t + 1)$ of collinearity graph.

A set of points $S$ is tight if $E \chi_S = 0$.

- For $S \neq \emptyset$, we have $\sum_{i=0}^{d} a_i/(-s)^i \geq 0$, with equality iff $S$ is tight.

Properties of tight sets

- For every point $p$: $\sum_{i=0}^{d} B_{p,i}/(-s)^i = 0$
  ($B_{p,i}$ is number of points in $S$ at distance $i$ from $p$)
- $|S| = i(s + 1)$ for some integer $i$ (we say $S$ is $i$-tight).
- Any union of $m$ disjoint lines is $m$-tight.
- 1-tight set = set of $s + 1$ points on some line.
Consider a regular near 2d-gon of order \((s, t)\).

**Tight subsets of points**

Let \(E\) be idempotent corresponding to eigenvalue \(-(t + 1)\) of collinearity graph.

A set of points \(S\) is *tight* if \(E \chi_S = 0\).

- For \(S \neq \emptyset\), we have \(\sum_{i=0}^{d} a_i/(-s)^i \geq 0\), with equality iff \(S\) is tight.

**Properties of tight sets**

- For every point \(p\): \(\sum_{i=0}^{d} B_{p,i}/(-s)^i = 0\)  
  \((B_{p,i} \text{ is number of points in } S \text{ at distance } i \text{ from } p)\)

- \(|S| = i(s + 1)\) for some integer \(i\) (we say \(S\) is \(i\)-tight).

- Any union of \(m\) disjoint lines is \(m\)-tight.

- 1-tight set = set of \(s + 1\) points on some line.

Payne (1987): introduced tight sets for regular near 4-gons.
Consider a regular near $2d$-gon of order $(s, t)$.
Let $E_0$ be trivial idempotent, $E$ be idempotent for eigenvalue $-(t + 1)$.

$m$-ovoids

$S$ is an $m$-ovoid if every line intersects $S$ in exactly $m$ points.
Since $\text{Im}(E)$ is kernel of incidence matrix:

$$S \text{ is } m\text{-ovoid } \iff \chi_S = E_0\chi_S + E\chi_S.$$
Consider a regular near $2d$-gon of order $(s, t)$. Let $E_0$ be trivial idempotent, $E$ be idempotent for eigenvalue $-(t + 1)$.

### $m$-ovoids

$S$ is an $m$-ovoid if every line intersects $S$ in exactly $m$ points. Since $\text{Im}(E)$ is kernel of incidence matrix:

$$S \text{ is } m\text{-ovoid} \iff \chi_S = E_0\chi_S + E\chi_S.$$  

### Tight set versus $m$-ovoid

An $i$-tight set $S_1$ is design-orthogonal to $m$-ovoid $S_2$ with $|S_1 \cap S_2| = mi$.

Consider generalized 6-gon of order \((s, t)\) (so \(c_2 = 1\)).
If \(t = s^3, s > 1 \implies s(s^3 + 1), -s^3 - 1, s^2 + s - 1, -s^2 + s - 1\) is
\(Q\)-polynomial ordering eigenvalues of collinearity graph
Consider generalized 6-gon of order \((s, t)\) (so \(c_2 = 1\)).

If \(t = s^3, s > 1 \implies s(s^3 + 1), -s^3 - 1, s^2 + s - 1, -s^2 + s - 1\) is a \(Q\)-polynomial ordering eigenvalues of collinearity graph.

**Intersecting 1-ovoids \(S_1, S_2\) for order \((s, s^3)\)**

- 1-ovoid: coclique s.t. every line contains 1 point.
- Two 1-ovoids \(S_1, S_2\) satisfy: \(\chi_{S_1}, \chi_{S_2}\) in span of first 2 eigenspaces.
- \(\chi_{S_1 \cap S_2}\) orthogonal to last eigenspace, so every \(p \in S_1 \cap S_2\) satisfies:
  \[
  B_{p,\cdot} = \left(1, 0, \frac{(s+1)|S_1 \cap S_2|}{s^2 + s + 1} + (s^2-1)(s^3+1), s^2\left(\frac{|S_1 \cap S_2|}{s^2 + s + 1} - (s^3-s+1)\right)\right).
  \]
- Hence \(|S_1 \cap S_2| = 0\) or \(h(s^2 + s + 1)\) for some integer \(h \geq s^3 - s + 1\).
Consider generalized 6-gon of order \((s, t)\) (so \(c_2 = 1\)).
If \(t = s^3, s > 1 \implies s(s^3 + 1), -s^3 - 1, s^2 + s - 1, -s^2 + s - 1\) is
\(Q\)-polynomial ordering eigenvalues of collinearity graph

**Intersecting 1-ovoids** \(S_1, S_2\) for order \((s, s^3)\)

- 1-ovoid: coclique s.t. every line contains 1 point
- Two 1-ovoids \(S_1, S_2\) satisfy: \(\chi_{S_1}, \chi_{S_2}\) in span of first 2 eigenspaces
- \(\chi_{S_1} \cap S_2\) orthogonal to last eigenspace, so every \(p \in S_1 \cap S_2\) satisfies:

\[
B_{p,.} = \left(1, 0, \frac{(s + 1)|S_1 \cap S_2|}{s^2 + s + 1} + (s^2 - 1)(s^3 + 1), s^2\left(\frac{|S_1 \cap S_2|}{s^2 + s + 1} - (s^3 - s + 1)\right)\right).
\]

- Hence \(|S_1 \cap S_2| = 0\) or \(h(s^2 + s + 1)\) for some integer \(h \geq s^3 - s + 1\).

**Examples?**

Only known generalized hexagons of order \((s, s^3), s > 1\), are
*dual twisted-triality hexagons* \(T(q, q^3)\) with \(q\) prime power.

De Wispelaere-Van Maldeghem (2004): no 1-ovoids in \(T(2, 8), T(3, 27)\).
Consider generalized octagon of order \((s, t), s > 1\) (so \(c_2 = c_3 = 1\)).

**Suboctagons**

- Let \(S\) be point-set of suboctagon of order \((s', t')\).
- Inner distribution \(a\) consists of valencies suboctagon.
- Use idempotent for eigenvalue \(-(t + 1)\):

\[
\sum_{i=0}^{4} \frac{a_i}{(-s)^i} = \frac{(s - s')(s - s't')(s^2 + (s't')^2)}{s^4} \geq 0,
\]

with equality iff \(S\) is tight.
- Thas (1979): \((s = s') \implies t' = 1\), so \(s \geq s't'\) holds in any case.
Consider vector space with non-degenerate form with Witt index $d$. Points dual polar space = totally isotropic $d$-spaces

**Eigenspaces**

- $Q$-polynomial ordering of idempotents $E_0, \ldots, E_d$ ("regular semilattice ordering"), $E_d$ corresponds to $\lambda_d = -(t + 1)$.
- Point-set $S$ with $\chi_S = E_0\chi_S + (E_{t+1}\chi_S + \ldots + E_d\chi_S)$ is $t$-design: behaving nicely with respect to totally isotropic $t$-spaces
Consider vector space with non-degenerate form with Witt index $d$. Points dual polar space = totally isotropic $d$-spaces

**Eigenspaces**

- $Q$-polynomial ordering of idempotents $E_0, \ldots, E_d$ ("regular semilattice ordering"), $E_d$ corresponds to $\lambda_d = -(t + 1)$.
- Point-set $S$ with $\chi_S = E_0\chi_S + (E_{t+1}\chi_S + \ldots + E_d\chi_S)$ is $t$-design: behaving nicely with respect to totally isotropic $t$-spaces

The case $\chi_S = E_0\chi_S + E_1\chi_S + E_d\chi_S$

- *dual zero intervals*, see Suda (2011)
- $S$ is "nicer" with respect to $(d-1)$-spaces than to $d$-spaces
- Similar ideas by Calderbank-Delsarte (1993) and Delsarte (2004)
- Application: Pepe-Storme-V. (2011): extremal Erdős-Ko-Rado subsets in $C_d(q)/W(2d-1, q)$ for odd $d$. (descendents in Ustimenko graph, see Tanaka (201?))
Polar space from Hermitian form on $V(2d, q^2)$: $^2A_{2d-1}(q)/H(2d - 1, q^2)$. Consider subset $S$ of totally isotropic $d$-spaces.

$m$-ovoids

- Here: $S$ is $m$-ovoid iff $\chi_S = E_0\chi_S + E_d\chi_S$
  (i. e. every $(d - 1)$-space in exactly $m$ elements of $S$).
Polar space from Hermitian form on $V(2d, q^2): {^2A_{2d-1}}(q)/H(2d - 1, q^2)$. Consider subset $S$ of totally isotropic $d$-spaces.

$m$-ovoids

- Here: $S$ is $m$-ovoid iff $\chi_S = E_0 \chi_S + E_d \chi_S$
  (i. e. every $(d - 1)$-space in exactly $m$ elements of $S$).
- Segre (1965): non-trivial $S$ must be half of set of $d$-spaces: $m = (q + 1)/2$
- V. (2011): if $S$ is such a half, it induces distance-regular graph with classical parameters $(d, -q, -(q + 1)/2, -((-q)^d + 1)/2)$.
- Cossidente-Penttila (2005): construction of such “hemisystems” for odd $q$ and $d = 2$.
- Open problem for $d \geq 3$!
Special property of $^2A_{2d-1}(q)/H(2d - 1, q^2)$: two $Q$-polynomial orderings!

The case $d = 3$

- orderings: $E_0, E_1, E_2, E_3$ (semilattices) and $E_0, E_3, E_1, E_2$. 
Special property of $^2A_{2d-1}(q)/H(2d-1,q^2)$: two $Q$-polynomial orderings!

The case $d = 3$

- orderings: $E_0, E_1, E_2, E_3$ (semilattices) and $E_0, E_3, E_1, E_2$.
- If $S_1$ and $S_2$ are $(q + 1)/2$-ovoids, then 2nd ordering implies:

$$\chi_{S_1 \cap S_2} = E_0 \chi_{S_1 \cap S_2} + E_3 \chi_{S_1 \cap S_2} + E_1 \chi_{S_1 \cap S_2}$$

$$= (E_0 \chi_{S_1 \cap S_2} + E_1 \chi_{S_1 \cap S_2}) + E_3 \chi_{S_1 \cap S_2},$$

so dual zero interval in 1st ordering!
Special property of $^2A_{2d-1}(q)/H(2d-1,q^2)$: two $Q$-polynomial orderings!

The case $d = 3$

- orderings: $E_0, E_1, E_2, E_3$ (semilattices) and $E_0, E_3, E_1, E_2$.
- If $S_1$ and $S_2$ are $(q + 1)/2$-ovoids, then 2nd ordering implies:

$$
\chi S_1 \cap S_2 = E_0 \chi S_1 \cap S_2 + E_3 \chi S_1 \cap S_2 + E_1 \chi S_1 \cap S_2
$$

$$
= (E_0 \chi S_1 \cap S_2 + E_1 \chi S_1 \cap S_2) + E_3 \chi S_1 \cap S_2,
$$

so dual zero interval in 1st ordering!

- $\Rightarrow$ if line $\ell$ of dual polar space contains $x$ points of $S_1 \cap S_2$:

$$
x(q^3 - 1) + \frac{|S_1 \cap S_2|}{q^3 + 1},
$$

elements of $S_1 \cap S_2$ at distance 1 from $\ell$ in the near hexagon.

$\Rightarrow |S_1 \cap S_2|$ is divisible by $q^3 + 1$. 

Frédéric Vanhove (UGent)
Thank you for your attention!

Slides (and more) on http://cage.ugent.be/~fvanhove