

# Extremal and regular subsets in regular near polygons

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## Outline

- Introduction: (regular) near  $2d$ -gons, association schemes, distance-regular graphs....
- Substructures in regular near  $2d$ -gons
- Generalized polygons
- Dual polar spaces

*Point-line geometry* is ordered triple  $(P, L, I)$ ,  $P, L \neq \emptyset, I \subseteq (P \times L)$ .

- If  $(p, \ell) \in I$  then  $p$  is on  $\ell$ , or  $\ell$  contains  $p$ .
- *Collinearity graph* has vertex set  $P$ , with 2 points adjacent when on common line.

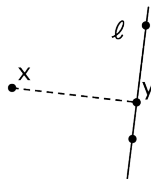
## Definition

Near  $2d$ -gons,  $d \geq 2$ , are point-line geometries:

- 1 2 points are on at most 1 line  
(every line contains at least 2 points),
- 2 the collinearity graph on points has diameter  $d$ ,

$\forall$  point  $x$  and  $\forall$  line  $\ell$ ,

- 3 there is unique point  $y \in \ell$   
at min. distance from  $x$ .



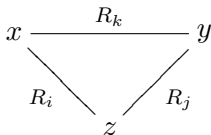
Simplest example: ordinary  $2d$ -gon!

We will want high regularity!

## Association schemes

$(\Omega, \{R_0, \dots, R_d\})$ , with  $\Omega \neq \emptyset$  finite set, is association scheme if:

- $\{R_0, \dots, R_d\}$  partitions  $\Omega \times \Omega$ ,
- $R_0$  is identity relation,
- $(\omega_1, \omega_2) \in R_i \iff (\omega_2, \omega_1) \in R_i$ ,
- there are *intersection numbers*  $p_{ij}^k$ :  
if  $(x, y) \in R_k$ , the number of elements  $z$  in  $\Omega$   
for which  $(x, z) \in R_i$  and  $(z, y) \in R_j$  is  $p_{ij}^k$ .



## Definition of matrices $A_i$

Consider association scheme  $(\Omega, \{R_0, \dots, R_d\})$

and order the elements of  $\Omega$ :  $\omega_1, \dots, \omega_{|\Omega|}$ .

For each  $R_i$ , define real  $(|\Omega| \times |\Omega|)$ -matrix  $A_i$ :

$$\begin{cases} (A_i)_{rs} &= 1 \text{ if } (\omega_r, \omega_s) \in R_i, \\ (A_i)_{rs} &= 0 \text{ if } (\omega_r, \omega_s) \notin R_i. \end{cases}$$

## Properties

- $A_0 + \dots + A_d$  is all-one matrix.
- $A_0$  is identity matrix.
- $A_i$  is symmetric.
- $A_i A_j = \sum_k p_{ij}^k A_k$ .

*Bose-Mesner algebra*: algebra with basis  $\{A_0, \dots, A_d\}$ .

## Eigenvectors for scheme $(\Omega, \{R_0, \dots, R_d\})$

$\mathbb{R}^\Omega$ : real vector space with orthonormal basis indexed by elements of  $\Omega$ .

$\mathbb{R}^\Omega$  uniquely decomposes as:

$$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \dots \perp V_d,$$

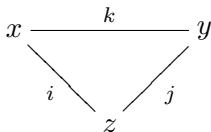
with every  $v \neq 0 \in V_j$  an eigenvector of every relation  $R_i$ :  $A_i v = \lambda_{ji} v$ .

## Idempotents

- Orthogonal projection  $E_j$  onto  $V_j$  is also in  $\langle A_0, \dots, A_d \rangle$ .
- These *minimal idempotents* form second basis  $\{E_0, \dots, E_d\}$  (we let  $E_0$  denote projection onto all-one vector).

## Distance-regular graphs

- Consider graph  $\Gamma = (\Omega, E)$  of diameter  $d$ .
- Write  $\Gamma_i(x)$  for vertices at distance  $i$  from  $x$ .
- Define  $R_i$  as  $\{(x, y) \in \Omega \times \Omega \mid d(x, y) = i\}$ .
- $\Gamma$  is *distance-regular* if  $(\Omega, \{R_0, \dots, R_d\})$  is association scheme.



So number of  $z$  in  $\Gamma_i(x) \cap \Gamma_j(y)$  must be constant  $p_{ij}^k$ .

Convention:

$$c_k = |\Gamma_1(x) \cap \Gamma_{k-1}(y)|, a_k = |\Gamma_1(x) \cap \Gamma_k(y)|, b_k = |\Gamma_1(x) \cap \Gamma_{k+1}(y)|.$$

Such association schemes are called *P-polynomial* or *metric*.

## $Q$ -polynomial association schemes

- Bose-Mesner algebra  $\langle A_0, \dots, A_d \rangle = \langle E_0, \dots, E_d \rangle$  is also closed under entrywise multiplication “ $\circ$ ”.
- $E_0, \dots, E_d$  is a  $Q$ -polynomial or *cometric* ordering if:

$$|\Omega|(E_1 \circ E_i) = b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1},$$

with  $c_i^* \neq 0$  for  $i \in \{1, \dots, d\}$ .

Recall: Near  $2d$ -gons,  $d \geq 2$ , are point-line geometries:

- 1 2 points on at most one line (every line contains at least 2 points),
- 2 the collinearity graph on points has diameter  $d$ ,
- 3  $\forall$  point  $x$  and  $\forall$  line  $\ell$ , there is unique point  $y \in \ell$  at minimal distance from  $x$ .

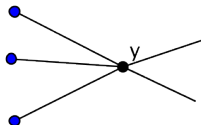
Near  $2d$ -gon is *regular* if the collinearity graph is distance-regular.

$\implies$  Then it has an *order*  $(s, t)$ ,  $s, t \geq 1$ :

$s + 1$  points on each line,  $t + 1$  lines through each point.

If  $d(x, y) = i$  then through  $y$ :

- $c_i$  lines with 1 point at distance  $i - 1$  from  $x$ , and its  $s$  other points at distance  $i$  from  $x$ ,
- $(t + 1) - c_i$  lines at distance  $i$  from  $x$ .



Some types of regular near  $2d$ -gons of order  $(s, t)$ :

### Generalized $2d$ -gons

For every point  $p$  and line  $\ell$ , there is a unique shortest path from  $p$  to  $\ell$ .  
Here  $c_1 = \dots = c_{d-1} = 1$  and  $c_d = t + 1$ .

### Dual polar spaces

Consider vector space with non-degenerate quadratic/alternating/Hermitian form of Witt index  $d$ .

*Dual polar space:*

- “points”: totally isotropic  $d$ -spaces
- “line”: totally isotropic  $(d - 1)$ -spaces

Here  $c_i = (q^i - 1)/(q - 1)$  and  $s = q^e$ .

Consider association scheme  $(\Omega, \{R_0, \dots, R_d\})$  and non-empty  $S \subseteq \Omega$ .

### Counting with respect to $S$

- inner distribution  $\mathbf{a}$ :  $\mathbf{a}_i = \frac{|R_i \cap (S \times S)|}{|S|}$  (“average inner valency”)
- outer distribution  $B$ :  $B_{\omega,i} = |R_i \cap (\{\omega\} \times S)|$
- characteristic vector of subset  $S \subseteq \Omega$ :

$$\chi_S = (1, 1, 0, \dots, 1, 0)^T \in \mathbb{R}^\Omega,$$

Consider minimal idempotent  $E = (\theta_0 A_0 + \dots + \theta_d A_d) / |\Omega|$

Delsarte (1973)

$$\theta_0 \mathbf{a}_0 + \dots + \theta_d \mathbf{a}_d \geq 0,$$

with equality iff  $E\chi_S = 0$  (i.e. iff  $\chi_S \in \text{Im}(E)^\perp$ ), and then:

$$\theta_0 B_{\omega,0} + \dots + \theta_d B_{\omega,d} = 0, \forall \omega \in \Omega.$$

Consider association scheme  $(\Omega, \{R_0, \dots, R_d\})$  and  $S_1, S_2 \subseteq \Omega$ .  
 What about  $S_1 \cap S_2$ ?

Delsarte (1977): Design-orthogonality

Observe:  $|S_1 \cap S_2| = (\chi_{S_1})^T \chi_{S_2}$ .

$S_1$  and  $S_2$  are *design-orthogonal* if  $\forall j \neq 0: E_j \chi_{S_1} = 0$  or  $E_j \chi_{S_2} = 0$ .  
 Then:  $|S_1 \cap S_2| = |S_1| |S_2| / |\Omega|$ .

Martin (2001): Vanishing Krein parameters  $q_{ij}^k$

Observe  $\chi_{S_1 \cap S_2} = \chi_{S_1} \circ \chi_{S_2}$  (entrywise product).

If  $(E_i \circ E_j) E_k = 0$  (i.e. if  $q_{ij}^k = 0$ ) then

$$E_k(u \circ v) = 0, \forall u \in \text{Im}(E_i), \forall v \in \text{Im}(E_j).$$

$\implies$  properties of  $\chi_{S_1 \cap S_2}$

Consider collinearity graph  $\Gamma$  of regular near  $2d$ -gon of order  $(s, t)$ .

### Eigenvalues of $\Gamma$

- Largest eigenvalue: valency  $s(t + 1)$ .  
Eigenspace: spanned by all-one vector  $\chi_\Omega$
- Smallest eigenvalue:  $-(t + 1)$ .  
Eigenspace: kernel incidence matrix between points and lines  
Corresponding idempotent is (up to positive scalar):

$$\frac{A_0}{1} + \frac{A_1}{-s} + \dots + \frac{A_d}{(-s)^d}.$$

Consider a regular near  $2d$ -gon of order  $(s, t)$ .

### Tight subsets of points

Let  $E$  be idempotent corresponding to eigenvalue  $-(t + 1)$  of collinearity graph.

A set of points  $S$  is *tight* if  $E\chi_S = 0$ .

- For  $S \neq \emptyset$ , we have  $\sum_{i=0}^d \mathbf{a}_i / (-s)^i \geq 0$ , with equality iff  $S$  is tight.

### Properties of tight sets

- For every point  $p$ :  $\sum_{i=0}^d B_{p,i} / (-s)^i = 0$   
( $B_{p,i}$  is number of points in  $S$  at distance  $i$  from  $p$ )
- $|S| = i(s + 1)$  for some integer  $i$  (we say  $S$  is  $i$ -tight).
- Any union of  $m$  disjoint lines is  $m$ -tight.
- 1-tight set = set of  $s + 1$  points on some line.

Payne (1987): introduced tight sets for regular near 4-gons.

Consider a regular near  $2d$ -gon of order  $(s, t)$ .

Let  $E_0$  be trivial idempotent,  $E$  be idempotent for eigenvalue  $-(t + 1)$ .

### $m$ -ovals

$S$  is an  $m$ -oval if every line intersects  $S$  in exactly  $m$  points.

Since  $\text{Im}(E)$  is kernel of incidence matrix:

$$S \text{ is } m\text{-oval} \iff \chi_S = E_0\chi_S + E\chi_S.$$

### Tight set versus $m$ -oval

An  $i$ -tight set  $S_1$  is design-orthogonal to  $m$ -oval  $S_2$  with

$$|S_1 \cap S_2| = mi.$$

Similar concepts/results for partial geometries, projective and polar spaces by Eisfeld (1998), Bamberg-Kelly-Law-Penttila (2007), De Wispelaere-Van Maldeghem(2008), Bamberg-Law-Penttila (2009)

Consider generalized 6-gon of order  $(s, t)$  (so  $c_2 = 1$ ).

If  $t = s^3, s > 1 \implies s(s^3 + 1), -s^3 - 1, s^2 + s - 1, -s^2 + s - 1$  is  $Q$ -polynomial ordering eigenvalues of collinearity graph

Intersecting 1-ovals  $S_1, S_2$  for order  $(s, s^3)$

- 1-ovoid: coclique s.t. every line contains 1 point
- Two 1-ovals  $S_1, S_2$  satisfy:  $\chi_{S_1}, \chi_{S_2}$  in span of first 2 eigenspaces
- $\chi_{S_1 \cap S_2}$  orthogonal to last eigenspace, so every  $p \in S_1 \cap S_2$  satisfies:

$$B_{p..} = \left( 1, 0, \frac{(s+1)|S_1 \cap S_2|}{s^2 + s + 1} + (s^2 - 1)(s^3 + 1), s^2 \left( \frac{|S_1 \cap S_2|}{s^2 + s + 1} - (s^3 - s + 1) \right) \right).$$

- Hence  $|S_1 \cap S_2| = 0$  or  $h(s^2 + s + 1)$  for some integer  $h \geq s^3 - s + 1$ .

Examples?

Only known generalized hexagons of order  $(s, s^3), s > 1$ , are *dual twisted-triality hexagons*  $T(q, q^3)$  with  $q$  prime power.

De Wispelaere-Van Maldeghem (2004): no 1-ovals in  $T(2, 8), T(3, 27)$ .

Consider generalized octagon of order  $(s, t)$ ,  $s > 1$  (so  $c_2 = c_3 = 1$ ).

## Suboctagons

- Let  $S$  be point-set of suboctagon of order  $(s', t')$ .
- Inner distribution  $\mathbf{a}$  consists of valencies suboctagon.
- Use idempotent for eigenvalue  $-(t + 1)$ :

$$\sum_{i=0}^4 \mathbf{a}_i / (-s)^i = \frac{(s - s')(s - s't')(s^2 + (s't')^2)}{s^4} \geq 0,$$

with equality iff  $S$  is tight.

- Thas (1979):  $(s = s') \implies t' = 1$ , so  $s \geq s't'$  holds in any case.

Consider vector space with non-degenerate form with Witt index  $d$ .  
 Points dual polar space = totally isotropic  $d$ -spaces

## Eigenspaces

- $Q$ -polynomial ordering of idempotents  $E_0, \dots, E_d$   
 (“regular semilattice ordering”),  $E_d$  corresponds to  $\lambda_d = -(t + 1)$ .
- Point-set  $S$  with  $\chi_S = E_0\chi_S + (E_{t+1}\chi_S + \dots + E_d\chi_S)$  is  $t$ -design:  
 behaving nicely with respect to totally isotropic  $t$ -spaces

The case  $\chi_S = E_0\chi_S + E_1\chi_S + E_d\chi_S$

- *dual zero intervals*, see Suda (2011)
- $S$  is “nicer” with respect to  $(d - 1)$ -spaces than to  $d$ -spaces
- Similar ideas by Calderbank-Delsarte (1993) and Delsarte (2004)
- Application: Pepe-Storme-V. (2011):  
 extremal Erdős-Ko-Rado subsets in  $C_d(q)/W(2d - 1, q)$  for odd  $d$ .  
 (descendants in Ustimenko graph, see Tanaka (201?))

Polar space from Hermitian form on  $V(2d, q^2)$ :  ${}^2A_{2d-1}(q)/H(2d-1, q^2)$ .  
 Consider subset  $S$  of totally isotropic  $d$ -spaces.

### $m$ -ovoids

- Here:  $S$  is  $m$ -ovoid iff  $\chi_S = E_0\chi_S + E_d\chi_S$   
 (i. e. every  $(d-1)$ -space in exactly  $m$  elements of  $S$ ).
- Segre (1965): non-trivial  $S$  must be half of set of  $d$ -spaces:  
 $m = (q+1)/2$
- V. (2011): if  $S$  is such a half, it induces distance-regular graph  
 with classical parameters  $(d, -q, -(q+1)/2, -((-q)^d + 1)/2)$ .
- Cossidente-Penttila (2005): construction of such “hemisystems”  
 for odd  $q$  and  $d = 2$ .
- Open problem for  $d \geq 3$ !

Special property of  ${}^2A_{2d-1}(q)/H(2d-1, q^2)$ :  
two  $Q$ -polynomial orderings!

The case  $d = 3$

- orderings:  $E_0, E_1, E_2, E_3$  (semilattices) and  $E_0, E_3, E_1, E_2$ .
- If  $S_1$  and  $S_2$  are  $(q+1)/2$ -ovals, then 2nd ordering implies:

$$\begin{aligned}\chi_{S_1 \cap S_2} &= E_0 \chi_{S_1 \cap S_2} + E_3 \chi_{S_1 \cap S_2} + E_1 \chi_{S_1 \cap S_2} \\ &= (E_0 \chi_{S_1 \cap S_2} + E_1 \chi_{S_1 \cap S_2}) + E_3 \chi_{S_1 \cap S_2},\end{aligned}$$

so dual zero interval in 1st ordering!

- $\implies$  if line  $\ell$  of dual polar space contains  $x$  points of  $S_1 \cap S_2$ :

$$x(q^3 - 1) + \frac{|S_1 \cap S_2|}{q^3 + 1},$$

elements of  $S_1 \cap S_2$  at distance 1 from  $\ell$  in the near hexagon.

$\implies |S_1 \cap S_2|$  is divisible by  $q^3 + 1$ .

# Thank you for your attention!

Slides (and more) on <http://cage.ugent.be/~fvanhove>