An algebraic approach to projective spaces

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Outline

- Introduction of decomposition of $\mathbb{R}\Omega$ into invariant subspaces under group action.
- Decomposition of real algebra over subspaces of projective spaces.
- Geometric meaning of algebraic properties and consequences.
- Some examples.

Real algebras over finite sets

- $\mathbb{R}\Omega$: the real algebra over $\Omega = \{\omega_1, \ldots, \omega_{|\Omega|}\}$: $\{\Sigma_{i=1}^{|\Omega|} c_i\omega_i | c_i \in \mathbb{R}\}$.

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- The set $\{\omega_1, \ldots, \omega_{|\Omega|}\}$ is a basis of $\mathbb{R}\Omega$.
- The *characteristic vector* of a subset $S \subseteq \Omega$ is the formal sum χ_S of all elements of S.
- We define an inner product $\langle ., . \rangle$ on $\mathbb{R}\Omega$ with $\{\omega_1, \ldots, \omega_{|\Omega|}\}$ as orthonormal basis.
- The inner product of χ_{S_1} and χ_{S_2} is then $|S_1 \cap S_2|$.
  Example: $S_1 = \{\omega_1, \omega_2, \omega_4, \omega_5\}$, $S_2 = \{\omega_2, \omega_3, \omega_4\}$:

$$\begin{array}{rcccccc}
\chi_{S_1} & = & 1.\omega_1+ & 1.\omega_2+ & 0\omega_3+ & 1.\omega_4+ & 1.\omega_5 \\
\chi_{S_2} & = & 0.\omega_1+ & 1.\omega_2+ & 1.\omega_3+ & 1.\omega_4+ & 0.\omega_5 \\
\langle \chi_{S_1}, \chi_{S_2} \rangle & = & 1.0+ & 1.1+ & 0.1+ & 1.1+ & 1.0 \\
 & = & 2 & & & &
\end{array}$$

- $\chi_\Omega = \omega_1 + \cdots + \omega_{|\Omega|}$ (this is the *all-one vector*).
- $\mathbb{R}\Omega$ decomposes orthogonally as $\mathbb{R}\Omega = \langle \chi_\Omega \rangle \perp \langle \chi_\Omega \rangle^\perp$.
- Suppose $S \subseteq \Omega$, with $\chi_S = c\chi_\Omega + v$, $v \in \langle \chi_\Omega \rangle^\perp$, then:

$$|S| = |S \cap \Omega| = \langle \chi_S, \chi_\Omega \rangle = \langle c\chi_\Omega + v, \chi_\Omega \rangle = c\langle \chi_\Omega, \chi_\Omega \rangle = c|\Omega|,$$

and hence: $\chi_S = \frac{|S|}{|\Omega|}\chi_\Omega + v$, $v \in \langle \chi_\Omega \rangle^\perp$.

Every relation $R \subseteq (\Omega_1 \times \Omega_2)$ defines a linear map: $\mathbb{R}\Omega_1 \to \mathbb{R}\Omega_2$:

$$R(\omega) := \sum_{(\omega,\omega')\in R} \omega' \quad , \forall \omega \in \Omega_1.$$

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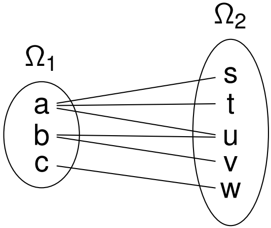
Example

$R = \{(a,s),(a,t),(a,u),(b,u),(b,v),(c,w)\} \subseteq (\Omega_1 \times \Omega_2)$.



- $R(a) = s + t + u, R(b) = u + v, R(c) = w$
- $R(\chi_{\{a,b\}}) = R(a+b) = s + t + 2u + v, R(b - 3c) = u + v - 3w$

Let (G, Ω) be a permutation representation (with G and Ω finite).

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- Every $g \in G$ also acts on $\mathbb{R}\Omega$: $(c_1\omega_1 + c_2\omega_2 + \cdots)^g = c_1(\omega_1^g) + c_2(\omega_2^g) + \cdots$.

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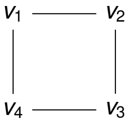
Theorem

If G acts generously transitively with rank $d + 1$ on Ω, then there is a unique decomposition: $\mathbb{R}\Omega = V_0 \oplus \ldots \oplus \ V_d$, with $V_i \neq 0$ and all V_i invariant (i.e. $V_i^g = V_i, \forall g \in G$). Moreover, this decomposition is orthogonal:$\mathbb{R}\Omega = V_0 \perp \cdots \perp \ V_d$, and $\langle \chi_\Omega \rangle = \langle \omega_1 + \cdots + \omega_{|\Omega|} \rangle$ is always one of the V_i.

Example: A square and its isometries

Take $\Omega = \{v_1, v_2, v_3, v_4\}$ and $G = D_8$.



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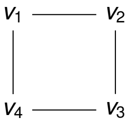

G acts generously transitively on Ω, with 3 symmetric orbits on $\Omega \times \Omega$:

$\{(v_1, v_1), (v_2, v_2), (v_3, v_3), (v_4, v_4)\}$,

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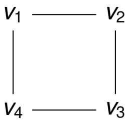

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$\mathbb{R}\Omega = V_0 \perp V_1 \perp V_2$, with:
$V_0 = \langle v_1 + v_2 + v_3 + v_4 \rangle = \langle \chi_\Omega \rangle$,
$V_1 = \langle v_1 + v_2 - v_3 - v_4, v_1 - v_2 - v_3 + v_4 \rangle$,
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Suppose G acts generously transitively on Ω with rank $d+1$, with decomposition $\mathbb{R}\Omega = \langle \chi_\Omega \rangle \perp V_1 \perp \ldots \perp V_d$ (so $(V_i)^g = V_i, \forall g \in G$). Let S and S' be subsets of Ω, such that χ_S and $\chi_{S'}$ have no components in the same $V_i, i > 0$.

$$\begin{aligned}
\chi_S &= \frac{|S|}{|\Omega|}\chi_\Omega + v_1 + 0 + v_3, \\
\chi_{S'} &= \frac{|S'|}{|\Omega|}\chi_\Omega + 0 + v_2' + 0, \\
\chi_{(S'^g)} = (\chi_{S'})^g &= \frac{|S'|}{|\Omega|}\chi_\Omega + 0 + (v_2')^g + 0. \\
|S \cap (S'^g)| = \langle \chi_S, \chi_{(S')^g} \rangle &= \frac{|S||S'|}{|\Omega|^2}\langle \chi_\Omega, \chi_\Omega \rangle + \langle v_1, 0 \rangle + \langle 0, (v_2')^g \rangle + \langle v_3, 0 \rangle, \\
&= |S||S'|/|\Omega|.
\end{aligned}$$

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Suppose G acts generously transitively on Ω with decomposition $\mathbb{R}\Omega = \langle \chi_\Omega \rangle \perp V_1 \perp \ldots \perp V_d$.
If $S, S' \subseteq \Omega$, then $|S \cap (S')^g|$ will be independent of $g \in G$ if and only if $\forall V_i \neq \langle \chi_\Omega \rangle$ either $\chi_S \in V_i^\perp$ or $\chi'_S \in V_i^\perp$.
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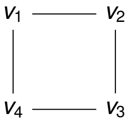
The main idea

A subset $S \subseteq \Omega$ is nice if its characteristic vector χ_S is in some $V_i^\perp$!

Example: The square revisited



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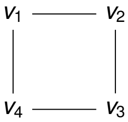

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$1/2(v_1 + v_2 + v_3 + v_4) + 1/2(v_1 + v_2 - v_3 - v_4) \in V_0 \perp V_1$.

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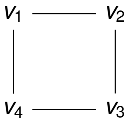

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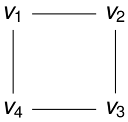

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$1/2(v_1 + v_2 + v_3 + v_4) + 1/2(v_1 - v_2 + v_3 - v_4) \in V_0 \perp V_2$.
Application of theorem:$|S \cap (S')^g| = |S||S'|/|\Omega| = 2.2/4 = 1, \forall g \in G$.
Verification: $(S')^g$ is either $\{v_1, v_3\}$ or $\{v_2, v_4\}, \forall g \in G$.

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- If $(\pi_1, \pi_2), (\pi'_1, \pi'_2) \in L_m \times L_m$, then:
  $(\exists g \in PGL(n+1, q) : (\pi_1, \pi_2)^g = (\pi'_1, \pi'_2)) \iff$
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Example: Lines in PG$(3, q)(n = 3, m = 1)$

- $\Omega = L_1$ and $G = PGL(4, q)$.
- 3 orbits on $L_1 \times L_1$:$\{(l_1, l_2) | l_1 = l_2\}$, $\{(l_1, l_2) | l_1 \text{and } l_2 \text{are concurrent}\}, \{(l_1, l_2) | l_1 \cap l_2 = \emptyset\}$.

The real algebras $\mathbb{R}L_m$ and the incidence maps

- We will consider the real algebras $\mathbb{R}L_m$ over *m*-spaces: $\mathbb{R}L_m = \{\sum_{(\pi_m)_i \in L_m} c_i(\pi_m)_i | c_i \in \mathbb{R}\}$
- With the incidence relation between m_1-spaces and m_2-spaces, we associate the *incidence map* $\iota_{m_1,m_2} : \mathbb{R}L_{m_1} \rightarrow \mathbb{R}L_{m_2}$:

$$\iota_{m_1,m_2}(\pi_{m_1}) = \sum_{\pi_{m_1} \mathrm{I} \pi_{m_2}} \pi_{m_2}, \forall \pi_{m_1} \in L_{m_1}.$$

Eisfeld (1999): structure of decomposition of $\mathbb{R}L_m$.

Example: PG$(4, q)$

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	
		V_1	V_1		

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V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	
		V_1	V_1		

Properties

- $V_{-1} = \langle \chi_\Omega \rangle$ (the 1-dimensional space, spanned by all-one-vector),
- If there is a V_i in both $\mathbb{R}L_{m_1}$ and $\mathbb{R}L_{m_2}$, then the incidence map ι_{m_1,m_2} is a bijection between both.
- If there is a V_i in $\mathbb{R}L_{m_1}$ and not in $\mathbb{R}L_{m_2}$,then ι_{m_1,m_2} vanishes on V_i.

- Consider a fixed m_1-space π_{m_1} in PG(n, q).
- The set S of all m_2-spaces in PG(n, q), incident with π_{m_1}, is given by: $\chi_S = \iota_{m_1,m_2}(\chi_{\{\pi_{m_1}\}})$.
- $\chi_S \in \mathbb{R}L_{m_2}$ will be in $V_i^\perp$ iff V_i appears in $\mathbb{R}L_{m_2}$, and not in $\mathbb{R}L_{m_1}$.

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Example: $\mathrm{PG}(5, q)$

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
			V_2			

If p is a point in $\mathrm{PG}(5, q)$, and S is the set of planes through p, then $\chi_S \notin V_{-1}^\perp, \notin V_0^\perp, \in V_1^\perp, \in V_2^\perp$

χ_S if S is set of points, lines, planes, 3-spaces, 4-spaces, 5-spaces through point π_0 in PG$(6, q)$:

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$	$\mathbb{R}L_6$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1	V_1		
			V_2	V_2			

χ_S if S is set of points, lines, planes, 3-spaces, 4-spaces, 5-spaces through point π_0 in PG$(6, q)$:

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$	$\mathbb{R}L_6$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1	V_1		
			V_2	V_2			

More generally: χ_S if S is set of points, lines, planes, 3-spaces, 4-spaces, 5-spaces through point π_0,in 5-space π_5 in PG$(6, q)$ with $\pi_0 \mathrm{I} \pi_5$:

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$	$\mathbb{R}L_6$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1	V_1		
			V_2	V_2			

In general: if S is set of m-spaces in PG(n, q) through π_{m_1} and in π_{m_2} $(m_1 \leq m \leq m_2)$ then $\chi_S \in V_i^{\perp}, \forall i \geq m_1 + (n - m_2) + 1$.

- Let S' be the set of m_2-spaces, incident with an m_1-space π_{m_1}.
- $\forall g \in PGL(n+1, q)$: S'^g is the set of m_2-spaces, incident with $(\pi_{m_1})^g$.

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Theorem (Repetition)

Suppose G acts generously transitively on Ω with decomposition $\mathbb{R}\Omega = \langle \chi_\Omega \rangle \perp V_1 \perp \ldots \perp V_d$.
If $S, S' \subseteq \Omega$, then $|S \cap (S')^g|$ will be independent of $g \in G$ if and only if $\forall V_i \neq \langle \chi_\Omega \rangle$ either $\chi_S \in V_i^\perp$ or $\chi'_S \in V_i^\perp$.
In that case: $|S \cap (S')^g| = \frac{|S||S'|}{|\Omega|}, \forall g \in G$.

Hence a set S of m_2-spaces will be such that every m_1-space is incident with a constant number of elements of S, iff $\chi_S \in V_i^\perp$ for all V_i appearing in $\mathbb{R}L_{m_1}$, except V_{-1}.

Example in $\mathrm{PG}(5, q)$

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
			V_2			

- A set of solids (or 3-spaces) S is such that every point (or 0-space) is covered a constant number of times by elements of S iff $\chi_S \in V_{-1} \perp V_1$.

Example in $\mathrm{PG}(5, q)$

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
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V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
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A set of planes (or 2-spaces) S is such that every line (or 1-space) is covered a constant number of times by elements of S iff $\chi_S \in V_{-1} \perp V_2$.

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V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
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		V_1	V_1	V_1		
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V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
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V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
			V_2			

A set of planes (or 2-spaces) S is such that every line (or 1-space) is covered a constant number of times by elements of S iff $\chi_S \in V_{-1} \perp V_2$.

If $\min(a, n-1-a) \leq min(b, n-1-b)$ (hence if there are at most as many V_i for $\mathbb{R}L_a$ as for $\mathbb{R}L_b$) then there are no sets S of a-spaces in PG(n, q), such that every b-space is incident with a constant number of elements of S, except $\emptyset$ and L_a.

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Argument in PG$(5, q)$ for $a = 1, b = 2$

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
			V_2			

- In $\mathbb{R}L_1$, all V_i also appear in $\mathbb{R}L_2$ as well.
- Hence $\chi_S(\in \mathbb{R}L_1)$ would have to be in $V_{-1} = \langle \chi_\Omega \rangle$.
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V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
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Let S be a set of a-spaces in $\mathrm{PG}(n, q)$. Then every b-space will be incident with a constant number of elements of S, if and only if every $(n - 1 - b)$-space is incident with a constant number of elements of S.

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Argument in $\mathrm{PG}(5, q)$ for $a = 2, b = 1$

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
			V_2			

- Every 1-space is covered by a constant number of planes in S iff $\chi_S \in V_{-1} \perp V_2$.
- Every 3-space covers a constant number of planes in S iff $\chi_S \in V_{-1} \perp V_2$.

If S is a set of m_2-spaces in $\mathrm{PG}(n, q)$, such that every m_1-space $(m_1 \leq m_2)$ is covered by a constant number of elements of S, then for every $c \leq \min(m_1, n - m_2)$, the nr. of elements of S incident with both π_{m_1-c} and π_{n-c}, $\pi_{m_1-c} \mathrm{I} \pi_{n-c}$, is also a constant.

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Proof for $\mathrm{PG}(6,q)$ with $m_1 = 1, m_2 = 2, c = 1$: planes covering lines equally often are also incident with a point and 5-space equally often

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$	$\mathbb{R}L_6$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1	V_1		
			V_2	V_2			

- Every line is in a constant number of planes of S: $\chi_S \in (V_{-1} \perp V_2) \subseteq \mathbb{R}L_2$
- If S' is set of planes through point π_0 and in hyperplane π_5 ($\pi_0 \mathrm{I} \pi_5$): $\chi'_S \in (V_{-1} \perp V_0 \perp V_1) \rightarrow$ apply Theorem on constant $|S \cap (S')^g|$.

- Consider a symplectic polarity θ in $\mathrm{PG}(2n+1, q)$, and let S_m denote the set of absolute m-spaces ($\pi_m \subseteq (\pi_m)^\theta$ or $(\pi_m)^\theta \subseteq \pi_m$).

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- $\chi_{S_m} \in V_i^\perp$ if and only if i is even: $\chi_{S_0} \in V_{-1}, \chi_{S_1} \in V_{-1} \perp V_1, \chi_{S_2} \in V_{-1} \perp V_1, \ldots$

Example: symplectic polarities in PG(5, q)

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
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V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
			V_2			

Hence if S is a set of m-spaces in PG($2n+1, q$), the number of totally isotropic m-spaces in S is the same for every symplectic polarity iff $\chi_S \in V_{-1} \perp V_0 \perp V_2 \perp \ldots$.

- If S is set of m-spaces in PG(n, q): every point is covered equally often iff $\chi_S \in V_0^{\perp} = V_{-1} \perp V_1 \perp \ldots$.

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- There are many examples of such sets: for instance $(t-1)$-spreads in $\mathrm{PG}(et-1, q)$.

- If S is set of m-spaces in PG(n, q): every point is covered equally often iff $\chi_S \in V_0^\perp = V_{-1} \perp V_1 \perp \ldots$.
- There are many examples of such sets: for instance $(t-1)$-spreads in PG$(et-1, q)$.
- But what about non-trivial sets of m-spaces S such that every line is covered equally often ($\Longleftrightarrow \chi_S \in V_0^\perp \cap V_1^\perp = V_{-1} \perp V_2 \ldots$)?

In PG(5, 2), 3 non-isomorphic sets of planes S exists, with $|S| = 279$ and $\chi_S \in V_{-1} \perp V_2$.

$\mathbb{R}L_{-1}$	$\mathbb{R}L_0$	$\mathbb{R}L_1$	$\mathbb{R}L_2$	$\mathbb{R}L_3$	$\mathbb{R}L_4$	$\mathbb{R}L_5$
V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}	V_{-1}
	V_0	V_0	V_0	V_0	V_0	
		V_1	V_1	V_1		
			V_2			

Implications by preceeding results

- Every line is covered by the same number of planes in S: 3.
- χ_S is also in $V_0^\perp \implies$ every point is in 31 planes of S.
- For every point π_0 and hyperplane π_4 through π_0, the number of planes in S incident with both is 7.
- The number of totally isotropic planes in S w.r.t. any symplectic polarity is 27.

Thank you for your attention!