Inequalities for regular near polygons, with applications to \( m \)-ovoids

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February 6, 2012, IGB (Ghent, Belgium)
Outline

- Introduction: regular near polygons, distance-regular graphs,...
- The inequality: equality, characterizations,...
- $m$-ovoids in case of equality: properties, non-existence,...
Consider graph $\Gamma = (\Omega, E)$
($\Omega$: vertex set, $E$: set of edges (pairs of vertices))
($\Omega \neq \emptyset$, undirected, no loops or multiple edges)

- A path of length $k$ is a sequence $(x_0, \ldots, x_k)$ with every 2 successive vertices adjacent.
- Distance between 2 vertices $x, y$:
  length of shortest path $(x, \ldots, y)$. (denoted by $d(x, y)$).
- Diameter: maximum distance in graph.
- $\Gamma_i(x)$: set of vertices at distance $i$ from vertex $x$.

Distance-regular graphs (or drgs)

A connected graph $\Gamma = (\Omega, E)$ is distance-regular if for $d(x, y) = k$ number of $z$ with $d(x, z) = i, d(y, z) = j$ is parameter $p_{i,j}^k$ only depending on $i, j, k$. 

$$
\begin{array}{ccc}
  x & \overset{k}{\longrightarrow} & y \\
  \downarrow{i} & & \downarrow{j} \\
 & z & 
\end{array}
$$

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Distance-regularity is equivalent to:
There are constants \( a_i, b_i, c_i \), such that for \( d(x, y) = i \)

\[
|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i, |\Gamma_i(x) \cap \Gamma_1(y)| = a_i, |\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i.
\]

Some properties

- \( c_1, \ldots, c_d \) and \( b_0, \ldots, b_{d-1} \) already determine all parameters \( p_{ij}^k \),
- number of vertices at distance \( i \) from any vertex is constant \( k_i \).

Classical parameters (Brouwer-Cohen-Neumaier (1989))

drg \( \Gamma \) has \textit{classical parameters} \((d, b, \alpha, \beta)\) if:

\[
b_i = \left( \left[\begin{array}{c} d \\ 1 \end{array}\right]_b - \left[\begin{array}{c} i \\ 1 \end{array}\right]_b \right) \left( \beta - \alpha \left[\begin{array}{c} i \\ 1 \end{array}\right]_b \right),\]

\[
c_i = \left[\begin{array}{c} i \\ 1 \end{array}\right]_b \left( 1 + \alpha \left[\begin{array}{c} i-1 \\ 1 \end{array}\right]_b \right),
\]

with \( \left[\begin{array}{c} i \\ 1 \end{array}\right]_b = 1 + \ldots + b^{i-1} \).
**Point-line geometry** is ordered triple \((P, L, I)\), \(P, L, \neq \emptyset, I \subseteq (P \times L)\).

- If \((p, \ell) \in I\) then \(p\) is on \(\ell\), or \(\ell\) contains \(p\).
- **Collinearity graph** has vertex set \(P\),
  with 2 points adjacent when on common line.

**Definition (Shult-Yanushka (1980))**

Near 2\(d\)-gons, \(d \geq 2\), are point-line geometries:

1. 2 points are on at most 1 line
   (every line contains at least 2 points),
2. the collinearity graph on points has diameter \(d\),
3. \(\forall\) point \(x\) and \(\forall\) line \(\ell\),
   there is unique point \(y \in \ell\)
   at min. distance from \(x\).

Simplest example: ordinary 2\(d\)-gon!

We need high regularity!
Recall: Near $2d$-gons, $d \geq 2$, are point-line geometries:

1. 2 points on at most one line (every line contains at least 2 points),
2. the collinearity graph on points has diameter $d$,
3. $\forall$ point $x$ and $\forall$ line $\ell$, there is unique point $y \in \ell$ at minimal distance from $x$.

Near $2d$-gon is regular if the collinearity graph is distance-regular.

$\implies$ Then it has an order $(s, t)$, $s, t \geq 1$:

$s + 1$ points on each line, $t + 1$ lines through each point.

If $d(x, y) = i$ then through $y$:

- $c_i$ lines with 1 point at distance $i - 1$ from $x$,
- and its $s$ other points at distance $i$ from $x$,
- $(t + 1) - c_i$ lines at distance $i$ from $x$.

Here $s$ and $c_2, \ldots, c_d = t + 1$ determine all parameters $a_i, b_i, c_i$ and $p_{ij}^k$!
Generalized 2d-gons of order \((s, t)\) (Tits (1959))

\(\forall\) point \(p\) and line \(\ell\), there is a unique shortest path from \(p\) to \(\ell\).

- Here \(c_1 = \ldots = c_{d-1} = 1\) and \(c_d = t + 1\).
- Dual (switching points and lines) is generalized 2d-gon of order \((t, s)\).
- Feit-Higman (1964): if \(s, t > 1\) then \(d \in \{2, 3, 4\}\).
(Classical) dual polar spaces

Consider vector space $V(n, q)$ with non-degenerate quadratic/alternating/Hermitian form of Witt index $d$.

- “points”: totally isotropic $d$-spaces
- “lines”: totally isotropic $(d - 1)$-spaces
- incidence: reversed containment

Properties:

- Here $c_i = (q^i - 1)/(q - 1)$ and $s = q^e$.
- Two points at distance $i$ $\iff$ intersection has dimension $d - i$.
- Characterization by Cameron (1982).
Another example: near hexagon related to $M_{24}$

Consider unique $5 - (24, 8, 1)$ design (with automorphism group $M_{24}$)

- “points”: 759 blocks,
- “lines”: 3795 triples of disjoint blocks,
- incidence: containment.

Unique near hexagon with those parameters $(s, c_2, c_3) = (2, 3, 15)$ by Brouwer (1983)
Adjacency matrices

Consider distance-regular graph $\Gamma$ with finite vertex set $\Omega$: Adjacency matrix $A_i$: $(\Omega \times \Omega)$-matrix over $\mathbb{R}$:

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & \text{if } d(x,y) \neq i \end{cases}.$$ 

- $A_i$ is symmetric,
- $A_0$ is identity matrix,
- $A_0 + \cdots + A_d$ is all-one matrix,
- $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$. 

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Eigenvectors for distance-regular graph $\Gamma$

$\mathbb{R}^\Omega$ is a real vector space with basis indexed by elements of finite vertex set $\Omega$.

$\mathbb{R}^\Omega$ uniquely decomposes as:

$$\mathbb{R}^\Omega = V_0 \perp V_1 \perp \ldots \perp V_d,$$

with every $v \neq 0 \in V_j$ an eigenvector of every relation $A_i$: $A_i v = \lambda_{ji} v$.

Idempotents

- Orthogonal projection $E_j$ onto $V_j$ is also in $\langle A_0, \ldots, A_d \rangle$.
- These *minimal idempotents* $E_j$ form second basis $\{E_0, \ldots, E_d\}$ for $\langle A_0, \ldots, A_d \rangle$.

Characteristic vector of subset $S \subseteq \Omega$

$$\chi_S = (1, 1, 0, \ldots, 1, 0)^T \in \mathbb{R}^\Omega.$$
Consider collinearity graph $\Gamma$ of finite regular near $2d$-gon of order $(s, t)$.

**Eigenvalues of $\Gamma$**

- **Largest eigenvalue**: valency $s(t + 1)$. 
  Eigenspace: spanned by all-one vector $\chi_\Omega$

- **Smallest eigenvalue**: $-(t + 1)$. 
  Eigenspace: kernel incidence matrix between points and lines 
  Corresponding idempotent is (up to positive scalar):

\[
M = \frac{A_0}{1} + \frac{A_1}{-s} + \ldots + \frac{A_d}{(-s)^d}.
\]
Theorem
If $S$ is a finite regular near $2d$-gon, $d \geq 3$, of order $(s, t)$, $s \geq 2$, then:

$$\frac{(s^i - 1)(c_{i-1} - s^{i-2})}{s^{i-2} - 1} \leq c_i \leq \frac{(s^i + 1)(c_{i-1} + s^{i-2})}{s^{i-2} + 1}, \forall i \in \{3, \ldots, d\},$$

Suppose $x$ and $y$ are two points of $S$ at distance $i \in \{3, 4, \ldots, d\}$, and put $Z := \Gamma_1(x) \cap \Gamma_{i-1}(y)$ and $Z' := \Gamma_{i-1}(x) \cap \Gamma_1(y)$.

1. If $c_i = (s^i - (-1)^i)(c_{i-1} - (-1)^i s^{i-2})/(s^{i-2} - (-1)^i)$ then $Mv = 0$ where $v = s(c_{i-1} - (-1)^i s^{i-2})(\chi_{\{x\}} - \chi_{\{y\}}) + (\chi_Z - \chi_{Z'})$.

2. If $c_i = (s^i + (-1)^i)(c_{i-1} + (-1)^i s^{i-2})/(s^{i-2} + (-1)^i)$ then $Mv = 0$ where $v = s(c_{i-1} + (-1)^i s^{i-2})(\chi_{\{x\}} + \chi_{\{y\}}) + (\chi_Z + \chi_{Z'})$. 

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Characterization w.r.t. to lower bound $c_3$

For a finite regular near $2d$-gon $S$, $d \geq 3$, of order $(s, t)$, $s \geq 2$, the following are equivalent:

1. $c_3$ attains lower bound $(s^2 + s + 1)(c_2 - s)$,
2. $c_i$ attains lower bound $(s^i - 1)(c_{i-1} - s^{i-2})/(s^{i-2} - 1)$, $\forall i \in \{3, \ldots, d\}$,
3. $s$ is a prime power and $S$ is dual polar space on $^2A_{2d-1}(q)/H(2d - 1, q^2)$, on $B_d(q)/Q(2d, q)$ or on $C_d(q)/W(2d - 1, q)$. 
Characterization (Terwilliger (1995))

For a finite regular near 2d-gon $S$, $d \geq 3$, of order $(s, t)$, $s \geq 2$, with collinearity graph $\Gamma$, the following are equivalent:

1. $c_3$ attains upper bound $(s^2 - s + 1)(c_2 + s)$,

2. $c_i$ attains upper bound $(s^i + 1)(c_{i-1} + s^{i-2})/(s^{i-2} + 1)$ for odd $i \geq 3$, and lower bound $(s^i - 1)(c_{i-1} - s^{i-2})/(s^{i-2} - 1)$ for even $i \geq 3$.

3. $\Gamma$ is “Q-polynomial” w.r.t. eigenvalue $-t - 1$,

4. $\Gamma$ has “classical parameters” $(d, -s, \alpha, \beta)$ for some $\alpha, \beta \in \mathbb{R}$. 
Consider a finite regular near $2d$-gon of order $(s, t)$.

**$m$-ovoids**

$\mathcal{O}$ is an $m$-void if every line intersects $\mathcal{O}$ in exactly $m$ points.

**Algebraic characterization**

Let $E_0$ be trivial idempotent, $E$ be idempotent for eigenvalue $-(t + 1)$. Since $\text{Im}(E)$ is kernel of incidence matrix:

$$
\mathcal{O} \text{ is } m\text{-void} \iff \chi_{\mathcal{O}} = E_0 \chi_{\mathcal{O}} + E \chi_{\mathcal{O}}.
$$

**m-ovoids in case of equality**

Consider finite regular near 2d-gon, $d \geq 3$, of order $(s, t)$, $s \geq 2$. Suppose $\mathcal{O}$ is an $m$-ovoid of $\mathcal{S}$.

Let $x$ and $y$ be points with $d(x, y) = i \geq 3$.

Put $N_x := |\Gamma_1(x) \cap \Gamma_{i-1}(y) \cap \mathcal{O}|$ and $N_y := |\Gamma_{i-1}(x) \cap \Gamma_1(y) \cap \mathcal{O}|$.

Suppose $c_i = (s^i - (-1)^i)(c_{i-1} - (-1)^i s^{i-2})/(s^{i-2} - (-1)^i)$.

1. If $x, y \in \mathcal{O}$: $N_x = N_y$.
2. If $x \in \mathcal{O}$, $y \notin \mathcal{O}$: $N_x - N_y = -s(c_{i-1} - (-1)^i s^{i-2})$.
3. If $x, y \notin \mathcal{O}$: $N_x = N_y$. 
**m-ovoids in case of equality**

Consider finite regular near $2d$-gon, $d \geq 3$, of order $(s, t)$, $s \geq 2$. Suppose $\mathcal{O}$ is an $m$-ovoid of $S$.

Let $x$ and $y$ be points with $d(x, y) = i \geq 3$.

Put $N_x := |\Gamma_1(x) \cap \Gamma_{i-1}(y) \cap \mathcal{O}|$ and $N_y := |\Gamma_{i-1}(x) \cap \Gamma_1(y) \cap \mathcal{O}|$.

Suppose $c_i = (s^i + (-1)^i)(c_{i-1} + (-1)^i s^{i-2})/(s^{i-2} + (-1)^i)$.

1. If $x, y \in \mathcal{O}$: $N_x + N_y = (c_{i-1} + (-1)^i s^{i-2}) \left(2m \frac{s^{i-1} + (-1)^i}{s^{i-2} + (-1)^i} - 2s\right)$.

2. If $x \in \mathcal{O}$, $y \notin \mathcal{O}$: $N_x + N_y = (c_{i-1} + (-1)^i s^{i-2}) \left(2m \frac{s^{i-1} + (-1)^i}{s^{i-2} + (-1)^i} - s\right)$.

3. If $x, y \notin \mathcal{O}$: $N_x + N_y = (c_{i-1} + (-1)^i s^{i-2}) \left(2m \frac{s^{i-1} + (-1)^i}{s^{i-2} + (-1)^i}\right)$. 

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Recall: 1-ovoid is set of points such that each line contains one element, i.e. coclique of size $|\mathcal{P}|/(s + 1)$.

**Theorem**

There are no 1-ovoids in dual polar space on $B_d(q)/Q(2d, q)$ or on $C_d(q)/W(2d − 1, q)$, if $d \geq 3$.

**Proof.**

Suppose $\mathcal{O}$ is 1-ovoid.
Consider $x, y \in \mathcal{O}$ with $d(x, y) = 3$ (this exists).
Consider $N_x = |\Gamma_1(x) \cap \Gamma_2(y) \cap \mathcal{O}|$ and $N_y = |\Gamma_2(x) \cap \Gamma_1(y) \cap \mathcal{O}|$.
Since $s = q$, $c_2 = q + 1$, $c_3 = (s^2 + s + 1)(c_2 − q) = q^2 + q + 1$:

$N_x + N_y = 2$ (with $i = 3, m = 1$) while $N_x = N_y = 0$. \(\square\)

Recall: 1-ovoid is set of points $\mathcal{O}$ such that each line contains one element, i.e. $\mathcal{O}$ coclique of size $|\mathcal{P}|/(s + 1)$.

So distance 1 is impossible within $\mathcal{O}$, while distances 2, 3 are possible in regular near hexagons!

**Theorem**

*Suppose $S$ is finite regular near hexagon of order $(s,t), s \geq 2$, with $c_3$ attaining upper bound $(s^2 - s + 1)(c_2 + s)$.*

*If $\mathcal{O}$ is 1-ovoid,*

then restriction distance-2-relation on $\mathcal{O}$ yields an srg($v,k,\lambda,\mu$):

\[
\begin{align*}
v &= (s^2 - s + 1)(s^2 + sc_2 + 1)(s^4 + s^3c_2 - s^3 - s^2c_2 + c_2)/c_2 \\
k &= s(s^2 - s + 1)(c_2 + s)(s^3 + s^2c_2 - s^2 - sc_2 + s + c_2 - 1)/c_2 \\
\lambda &= (s^2 - s + 1)(s^4 + 2s^3c_2 + c_2^2s^2 - s^2 - sc_2 - c_2)/c_2 \\
\mu &= (s^2 - s + 1)(s + c_2)(s^3 + s^2c_2 - s^2 - sc_2 + c_2)/c_2.
\end{align*}
\]
Feasibility of strongly regular graph
Consider 1-ovoid in finite regular near hexagon $S$ of order $(s, t), s \geq 2$, with $c_3 = t + 1 = (s^2 - s + 1)(c_2 + s)$.

Feasibility parameters srg & restriction ($c_2 - 1 | c_3 - 1$ or $c_2 = 1$) $\implies$ only these known possibilities:

- $s = 2, c_2 = 2, t = 11$:
  $S$ is near hexagon from ternary Golay code, 
  srg$(243, 132, 81, 60)$ is Delsarte graph (1972).

- $s = 2, c_2 = 3, t = 14$:
  $S$ is near hexagon from $5 - (24, 8, 1)$ design, 
  srg$(253, 140, 87, 65)$ is intersection-3 graph on 253 blocks of 
  $4 - (23, 7, 1)$ design.

- $s = 4, c_2 = 2, t = 77$:
  $S$ would have 235625 points 
  and collinearity graph with classical parameters $(3, -4, -5/3, 24)$, 
  1-ovoid would yield srg$(47125, 12012, 3575, 2886)$. 
Infeasibility for generalized hexagons of order \((s, s^3)\)

Generalized hexagons of order \((s, t)\)
are regular near hexagons with \(c_2 = 1\).
Here upper bound for \(c_3\) is \(s^3 + 1\) if \(s \geq 2\).
\(\Rightarrow\) 1-ovoid in finite generalized hexagon of order \((s, s^3), s \geq 2\),
would yield infeasible strongly regular graph!

Corollary

A finite generalized hexagon of order \((s, s^3), s \geq 2\), cannot be a proper
subhexagon of a (finite or infinite) generalized hexagon of order \((s, t)\).

Known generalized hexagons

Up to duality, we only know one (thick) finite generalized hexagon
\(H(q)\) of order \((q, q)\), and \(T(q, q^3)\) of order \((q, q^3)\), for a prime power \(q\).

- No 1-ovoids in \(T(2, 8)\) or \(T(3, 27)\),
- but 1-ovoids do exist in \(H(2), H(3), H(4)\).
Thank you for your attention!

Slides (and more) on http://cage.ugent.be/~fvanhove