Erdős-Ko-Rado Theorems for dual polar spaces

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The first Erdős-Ko-Rado Theorem

E.K.R. [1961]

If Ω is a set with n elements and S is a family of subsets of size k of Ω, with $n \geq 2k$, such that the elements of S are pairwise intersecting, then $|S| \leq \binom{n-1}{k-1}$.

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Characterization of the families of maximum size

If $|S| = \binom{n-1}{k-1}$, then:

- $2k < n$ and S is the family of subsets of size k containing a fixed element of Ω.
- $2k = n$ and S is either the family of subsets of size k containing a fixed element of Ω or it consists of the representatives of all the complementary pairs.

Analogue results

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B.M.I. Rands [1982]

The largest set of blocks of a $t-(v, k, \lambda)$ design pairwise intersecting has size equal to the number of blocks through a point and the blocks through a point is the only set of blocks meeting the bound, provided $v \geq f(k, t)$.

Analogue results

P.Frankl and R.M.Wilson [1986]/ C.D.Godsil and Newman [2006]

If V is a n–dimensional vector space over $\mathbb{F}_q$ and S is a family of k–dimensional subspaces of V pairwise intersecting non–trivially, with $n \geq 2k$, then $|S| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$. If $|S| = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$, then:

- $2k < n$ and S is the set of k–dimensional subspaces containing a fixed non–zero vector of V.
- $2k = n$ and S is either the set of k–dimensional subspaces containing a fixed non–zero vector of V or it is the set of k–dimensional subspaces of V contained in a hyperplane.

Graph theoretic approach

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Two vertices are adjacent iff their intersection is trivial.
A EKR set is a coclique of Γ.
If Γ is a v–regular graph with least eigenvalue τ and S is a coclique of Γ, then

$$|S| \leq \frac{|\Omega|}{1 - \frac{v}{\tau}}$$

and if $|S|$ meets the bound, then its characteristic vector χ_S is such that $\chi_S = \frac{|S|}{|\Omega|}\mathbf{1} + u$, where u is an eigenvector with eigenvalue τ.

Classical finite polar spaces

Classical finite polar spaces are incidence structures consisting of the lattices of subspaces of a finite projective space totally isotropic with respect to a certain non-degenerate sesquilinear form.

- the parabolic quadric $\mathcal{Q}(2n, q)$: $(n-1)$-dimensional generators,
- the hyperbolic quadric $\mathcal{Q}^+(2n+1, q)$: n-dimensional generators,
- the elliptic quadric $\mathcal{Q}^-(2n+1, q)$: $(n-1)$-dimensional generators,
- the symplectic space $W(2n+1, q)$: n-dimensional generators,
- the hermitian variety $\mathcal{H}(2n, q^2)$: $(n-1)$-dimensional generators,
- the hermitian variety $\mathcal{H}(2n+1, q^2)$: n-dimensional generators.

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We deal with the case of generators of polar spaces, when their dimension is at least two.

The bounds

Stanton [1980]:

Polar space	upper bound for $\|S\|$	Example of set meeting the bound
$\mathcal{Q}(2n,q)$	$\prod_{i=1}^{n-1}(q^i+1)$	generators through a point
$\mathcal{Q}^+(2n+1,q)$, n odd	$\prod_{i=0}^{n-1}(q^i+1)$	generators through a point
$\mathcal{Q}^+(2n+1,q)$, n even	$\prod_{i=1}^{n}(q^i+1)$	generators of one family
$\mathcal{Q}^-(2n+1,q)$	$\prod_{i=2}^{n}(q^i+1)$	generators through a point
$W(2n+1,q)$	$\prod_{i=1}^{n}(q^i+1)$	generators through a point
$\mathcal{H}(2n,q^2)$	$\prod_{i=1}^{n-1}(q^{2i+1}+1)$	generators through a point
$\mathcal{H}(2n+1,q^2)$, n odd	$\prod_{i=0}^{n-1}(q^{2i+1}+1)$	generators through a point
$\mathcal{H}(2n+1,q^2)$, n even	$\prod_{i=0,i\neq\frac{n}{2}}^{n}(q^{2i+1}+1)$	No examples known

Characterization of the sets meeting the bound

Our goal is to characterize the sets meeting the bounds.

- Is the point pencil the only possible construction for most of the polar spaces?
- For $\mathcal{Q}^+(2n+1, q)$, n even, are the generators of one family the only possible construction?
- What can we say about $\mathcal{H}(2n+1, q^2)$, n even?

Association schemes

A d-class *association scheme* on a finite set Ω is a pair $(\Omega, \mathcal{R})$ with $\mathcal{R}$ a set of symmetric relations $\{R_0, R_1, \ldots, R_d\}$ on Ω such that the following axioms hold:

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(i) R_0 is the identity relation,

(ii) $\mathcal{R}$ is a partition of Ω^2,

(iii) there are *intersection numbers* p_{ij}^k such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

All the relations R_i are symmetric regular relations with valency p_{ii}^0, and hence define regular graphs on Ω.

Association scheme on generators

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If the dimension of a generator is n, then on Ω we can define a set of n relations $\Gamma_i, i = 0, \cdots, n+1$ such that two generators are adjacent with respect to Γ_i iff they intersect in a space of codimension i. These relations give rise to an association scheme.

Fundamental results

Lemma

If S is a subset of Ω such that its characteristic vector $\chi_S = h\mathbf{1} + u$, where u is an eigenvector with eigenvalue λ for the adjacency matrix A_i of the relation Γ_i, then we have:

- every $p \in S$ has $\frac{|S|}{|\Omega|}(k - \lambda) + \lambda$ neighbors in S w.r.t. Γ_i
- every $p \notin S$ has $\frac{|S|}{|\Omega|}(k - \lambda)$ neighbors in S w.r.t Γ_i

where k is the valency of the graph Γ_i.

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where k is the valency of the graph Γ_i.

The number of neighbors of p depends only on the size of S

Most of the cases

For the following polar spaces:

- $\mathcal{Q}(2n, q)$, n even
- $\mathcal{Q}^-(2n+1, q)$
- $W(2n+1, q)$, n odd
- $\mathcal{H}(2n, q^2)$ and $\mathcal{H}(2n+1, q^2)$, n odd

if u is an eigenvector for the relation Γ_{n+1}, then it is a an eigenvector for $\Gamma_i, i = 0, \cdots, n$.

Most of the cases

For every *EKR* set S of maximum size, we know how many elements of S intersect a fixed generator π in a space of codimension i, $i = 1, \ldots, n$: this number is a constant and it does not depend on the geometric structure of the set S.

Known example of EKR in these polar spaces:

The generators through a fixed point.

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Known example of EKR in these polar spaces:

The generators through a fixed point.

For every $\pi \in S$, the number of elements of S intersecting π in a space of codimension i is the same as the point pencil construction.

We focus on a fixed a generator of S and we get:

Theorem

For the polar spaces $\mathcal{Q}(2n, q)$, n even, $\mathcal{Q}^-(2n+1, q)$, $W(2n+1, q)$, n odd, $\mathcal{H}(2n, q^2)$ and $\mathcal{H}(2n+1, q^2)$, n odd, the largest *EKR* set of generators is the set of generators through a fixed point.

Hyperbolic quadric $\mathcal{Q}^+(2n+1, q)$

In $\mathcal{Q}^+(2n+1, q)$ there are two system of generators, Ω_1 and Ω_2 of the same size, such that two generators π_1 and π_2 are in the same system iff $\dim \pi_1 \cap \pi_2$ has the same parity as n.

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Even n

The generators of Ω_i pairwise intersect in a non–empty space.
The size of Ω_i meets the Stanton bound.
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The size of Ω_i meets the Stanton bound.
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Odd n

If S is a maximum *EKR* set, then $S = S_1 \cup S_2$, where $S_i = S \cap \Omega_i$, $|S_1| = |S_2|$. If we find a *EKR* set S_i in $\Omega_i, i = 1,2$ and $|S_i| = \frac{|S|}{2}$, then $S_1 \cup S_2$ is a maximum *EKR* set in Ω.

$\mathcal{Q}^+(2n+1,q)$, n odd

We can focus on only one system of generators Ω_i.

Theorem

If $n > 3$ is odd, then S_i is the set of elements of Ω_i through a point. If $n = 3$, then S_i is either the set of elements of Ω_i through a point or it is the set of elements of Ω_i meeting a fixed element of Ω_j in a plane.

All generators: $n > 3$

We have two possibilities

- S is the set of all the generators through a point P
- S is the set of all the generators of one system through P_1 and the set of all the generators of the other system through P_2

$\mathcal{Q}^+(7,q)$

We have four possibilities

- S is the set of all the solids through a point P
- S is the set of all the solids of one system through P_1 and the set of all the solids of the other system through P_2
- S is the set of all solids of one system through P and all solids of the other system meeting Σ in a plane
- S is the set of all solids of one system meeting Σ_1 in a plane and all the generators meeting Σ_2 in a plane

Parabolic quadric $\mathcal{Q}(2n, q)$, n odd

Embed $\mathcal{Q}(2n, q)$, n odd, as a hyperplane section in a $\mathcal{Q}^+(2n+1, q)$: every generator of $\mathcal{Q}(2n, q)$ is contained in a unique generator of a fixed system Ω_i of $\mathcal{Q}^+(2n+1, q)$.

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An *EKR* set S of maximum size of $\mathcal{Q}(2n, q)$ gives rise to *EKR* set S' of maximum size of Ω_i.

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An *EKR* set S of maximum size of $\mathcal{Q}(2n, q)$ gives rise to *EKR* set S' of maximum size of Ω_i.

Theorem

Let $\mathcal{Q}(2n, q) = H \cap \mathcal{Q}^+(2n+1, q)$.
If $n > 3$, then S' is a point pencil and we have two possibilities:

- $P \in H$, so S is also a point pencil
- $P \notin H$, S is the set of generators of one system of a $\mathcal{Q}^+(2n-1, q)$ embedded in $\mathcal{Q}(2n, q)$.

If $n = 3$, then S' can be a point pencil or the generators meeting a fixed one in a plane, so we have a third possibility:

- S consists of the plane π and all the planes meeting π in a line

$W(2n+1,q)$, n and q even

If q is even, then:
$W(2n+1,q) \cong Q(2n+2,q)$
There is a $\mathcal{Q}^{+}(2n+1,q)$ inducing the symplectic polarity

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Theorem

An *EKR* set of maximum size S is

- a point pencil
- the set of generators of one system of a $\mathcal{Q}^+(2n+1,q)$
- $n=2$ and it consists of the plane π and the planes meeting π in a line

$W(2n+1,q)$, n even and q odd

Let $v_{\pi,S}$ be the vector of length n such that $(v_{\pi,S})_i$ is the number of elements of S meeting π in a space of codimension i, then:

$$v = hv_1 + (1-h)v_2$$

where

$W(2n+1, q)$, n even and q odd

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where v_1 arises from the point pencil construction

$W(2n+1, q)$, n even and q odd

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where v_1 arises from the point pencil construction and v_2 from the construction of the elements of one system of a hyperbolic quadric.

$W(2n+1, q)$, n even and q odd

Let $v_{\pi,S}$ be the vector of length n such that $(v_{\pi,S})_i$ is the number of elements of S meeting π in a space of codimension i, then:

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where v_1 arises from the point pencil construction and v_2 from the construction of the elements of one system of a hyperbolic quadric. Further investigation on the related association scheme and with more geometric arguments, we get:

Theorem

- S is a point pencil or
- $n = 2$ and S consists of the plane π and the planes meeting π in a line.

$\mathcal{H}(4n+1, q^2)$

Theorem

EKR set $|S| < \frac{|\Omega|}{1-\frac{k}{\tau}} = \frac{|\Omega|}{q^{2n+1}+1}$ (more than point-pencil).

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Theorem for planes in $\mathcal{H}(5, q^2)$

- maximum size: $1 + q + q^3 + q^5 < \frac{|\Omega|}{q^3+1} = (q+1)(q^5+1)$,
- only construction: a fixed plane and all the those meeting it in line.

If S is a point pencil, then $|S| = (q+1)(q^3+1) < 1 + q + q^3 + q^5$.

Polar space	EKR set of maximum size
$\mathcal{Q}(4n, q)$	point pencil
$\mathcal{Q}(4n+2, q) n \neq 2$	point pencil, generators of one system in a $\mathcal{Q}^+(4n+1, q)$
$\mathcal{Q}(6, q)$	point pencil, generators of one system in a $\mathcal{Q}^+(5, q)$ a fixed plane and the planes meeting it in a line
$\mathcal{Q}^+(4n+3, q)$, $n \neq 1$ a fixed system	point pencil
$\mathcal{Q}^+(7, q)$ a fixed system	point pencil solids meeting a fixed one of the other system in a plane
$\mathcal{Q}^+(4n+1, q)$	generators of one system
$\mathcal{Q}^-(2n+1, q)$	point pencil
$W(4n+3, q)$	point pencil
$W(4n+1, q) n \neq 1$	point pencil, generators of one system in $\mathcal{Q}^+(4n+1, q)$ q even
$W(5, q)$	point pencil, a fixed plane and the planes meeting it in a line generators of one system in $\mathcal{Q}^+(5, q)$ q even
$\mathcal{H}(2n, q^2), \mathcal{H}(4n+3, q^2)$	point pencil
$\mathcal{H}(5, q^2)$	a fixed plane and the planes meeting it in a line
$\mathcal{H}(4n+1, q^2) n > 1$	?