An algebraic approach to partial spreads of hermitian varieties

ULB-UGent-VUB-Seminar on Incidence Geometry

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$\mathcal{P}$	ϵ	d
$Q^-(2n+1,q)$	$+1$	$n-1$
$H(2n,q^2)$	$+1/2$	$n-1$
$Q(2n,q)$	0	$n-1$
$W(2n+1,q)$	0	n
$H(2n+1,q^2)$	$-1/2$	n
$Q^+(2n+1,q)$	-1	n

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- Number of points in $\mathcal{P}$: $(q^{d+1+\epsilon}+1)(q^{d+1}-1)/(q-1)$.
- *Generators* in a polar space $\mathcal{P}$= subspaces of max. projective dimension d.

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- When does a polar space $\mathcal{P}$ have a spread?
- If it doesn't, what size can its partial spreads still have?

Results on spreads in non-Hermitian polar spaces

$\mathcal{P}$	Existence of spreads
$Q^{-}(2n+1, q)$	q even: YES
$Q(2n, q)$	q odd, n even: NO; q even: YES
$W(2n+1, q)$	YES
$Q^{+}(2n+1, q)$	n even: NO; q even, n odd: YES

Hermitian variety $H(2n+1, q^2)$

Our main focus is on the Hermitian variety $H(2n+1, q^2)$:
Subspaces in $\mathrm{PG}(2n+1, q^2)$, all points $(X_0, \ldots, X_{2n+1})$ of which satisfy: $X_0^{q+1} + \cdots + X_{2n+1}^{q+1} = 0$.

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Properties

- $d = n$ and $\epsilon = -1/2$.
- Ovoid number: $q^{2n+1} + 1$.

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New result

We will give a generalisation of the last result...
by use of algebraic combinatorics!

Definition

D-class association scheme on finite set Ω:
Set of symmetric relations $\mathcal{R} = \{R_0, R_1, \ldots, R_D\}$ on Ω s.t.:

(i) R_0 is the identity relation,

(ii) $\mathcal{R}$ is a partition of Ω^2,

(iii) there are *intersection numbers* p_{ij}^k such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

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Example 1

Hamming scheme $H(n, q)$ on a set Γ of size q:

- Ω: Γ^n, i.e. all words of length n over Γ,
- $(\alpha, \beta) \in R_i \iff \alpha$ and β differ in i positions, $0 \leq i \leq n$.

Example 2: Three association schemes derived from $H(5, q^2)$

- Ω=Set of points of $H(5, q^2)$:
  - $(p, p') \in R_0 \iff p = p'$,
  - $(p, p') \in R_1 \iff \langle p, p' \rangle$ is a line of $H(5, q^2)$,
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- Ω=Set of lines of $H(5, q^2)$:
  - $(l, l') \in R_0 \iff l = l'$,
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  - $(l, l') \in R_2 \iff l \cap l' \neq \emptyset, \langle l, l' \rangle$ is not a plane of $H(5, q^2)$;
  - $(l, l') \in R_3 \iff l \cap l' = \emptyset, \langle l, l' \rangle$ contains a plane of $H(5, q^2)$,
  - $(l, l') \in R_4 \iff l \cap l' = \emptyset, \langle l, l' \rangle$ contains no plane of $H(5, q^2)$.

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  - $(\pi, \pi') \in R_0 \iff \pi = \pi'$,
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Real algebras over finite sets

- $\mathbb{R}\Omega$: the real algebra over $\Omega = \{\omega_1, \ldots, \omega_\Omega\}$: $\{\Sigma_{i=1}^{|\Omega|} c_i\omega_i | c_i \in \mathbb{R}\}$.

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$$R(\omega) := \sum_{(\omega,\omega')\in R} \omega' \quad , \forall \omega \in \Omega.$$

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- Matrix representation A of relation R with respect to basis $\{\omega_1, \ldots, \omega_{|\Omega|}\}$ of $\mathbb{R}\Omega$:

$$A_{ij} = \begin{cases} 1 \text{ if } (\omega_i, \omega_j) \in R, \\ 0 \text{ if } (\omega_i, \omega_j) \notin R. \end{cases}$$

R is a symmetric relation $\iff$ A is a symmetric matrix.

The Bose-Mesner algebra of an association scheme

$$\begin{array}{ccc} \vec{u} & \vec{v} & \vec{w} \end{array}$$
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

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  Hence: $R_i(R_j(v)) = R_j(R_i(v)) = \sum_{k=0}^{D} p_{ij}^k R_k(v), \forall v \in \mathbb{R}\Omega$.

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  Hence: $R_i(R_j(v)) = R_j(R_i(v)) = \sum_{k=0}^{D} p_{ij}^k R_k(v), \forall v \in \mathbb{R}\Omega$.
- The Bose-Mesner algebra of $\mathcal{R}$ is the algebra generated by $R_0, \ldots, R_D$.

Orthogonal decomposition of Bose-Mesner algebra

If $\mathcal{R}$ is a D-class association scheme on Ω, then there is a unique orthogonal decomposition $\mathbb{R}\Omega = V_0 \perp \ldots \perp V_D$, such that all V_i are eigenspaces of every relation R_j.

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Eigenvalues

The *matrix of eigenvalues* is a $(D+1) \times (D+1)$-matrix P, with P_{ij} the eigenvalue of R_j for V_i.

Example: Matrix of eigenvalues of Hamming scheme $H(3, q)$

$\Omega = \Gamma^3$ with $|\Gamma| = q$,
$(\alpha, \beta) \in R_i \iff \alpha$ and β differ in i positions, with $0 \leq i \leq 3$.

	R_0	R_1	R_2	R_3
V_0	1	$3(q-1)$	$3(q-1)^2$	$(q-1)^3$
V_1	1	$2q-3$	q^2-4q+3	$-(q-1)^2$
V_2	1	$q-3$	$-2q+3$	$q-1$
V_3	1	-3	3	-1

Distance-regular graphs

- Consider a connected graph Γ on V with diameter D.
- Define graphs Γ_i on V for every $i \in \{0, \ldots, D\}$:
  $v_1 \sim v_2 \iff d(v_1, v_2) = i$, and let R_i denote the corresponding symmetric relation on V.

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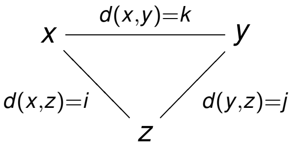
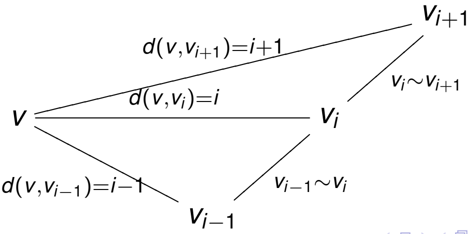
  $v_1 \sim v_2 \iff d(v_1, v_2) = i$, and let R_i denote the corresponding symmetric relation on V.
- Γ is distance-regular if $\{R_0, \ldots, R_D\}$ is an association scheme on V, or hence if:
  There are constants p_{ij}^k s.t. $\forall x, y$ with $d(x, y) = k$ there are p_{ij}^k vertices z with $d(x, z) = i$ and $d(z, y) = j$.



Weakening of axioms

A connected graph Γ is also distance-regular if there are constants k, b_i and c_i s.t.:

- Γ is regular with valency k.
- If $d(v, v_i) = i$, there are c_i neighbours v_{i-1} of v_i with $d(v, v_{i-1}) = i - 1, \quad \forall i \in \{1, \ldots, D\}$.
- If $d(v, v_i) = i$, there are b_i neighbours v_{i+1} of v_i with $d(v, v_{i+1}) = i + 1, \quad \forall i \in \{0, \ldots, D - 1\}$.



Sequences of eigenvalues

With every eigenvalue θ of Γ corresponds a sequence of eigenvalues $v_0, v_1, \ldots, v_D$ of $\Gamma_0, \Gamma_1, \ldots, \Gamma_D$, with $v_0 = 1, v_1 = \theta$ and

$$\theta v_i = c_{i+1} v_{i+1} + (k - b_i - c_i) v_i + b_{i-1} v_{i-1}, \quad \forall i \in \{1, \ldots, D-1\}.$$

Example: Distance-regular graph Γ on planes in $H(5, q^2)$

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- $\Gamma_0 : d(\pi, \pi') = 0 \iff \pi = \pi'$,
- $\Gamma_1 : d(\pi, \pi') = 1 \iff \pi \cap \pi'$ is a line,
- $\Gamma_2 : d(\pi, \pi') = 2 \iff \pi \cap \pi'$ is a point,
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Definition

Dual polar graph Γ on generators (or d-spaces) of polar space $\mathcal{P}$: two generators π_1, π_2 are adjacent iff $dim(\pi_1 \cap \pi_2) = d - 1$.

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Properties

- $d(\pi_1, \pi_2) = i$ if and only if $\pi_1 \cap \pi_2$ is a $(d - i)$-space.
- Γ is a distance-regular graph with diameter $d + 1$.
- In Γ_0, every generator is connected only to itself.
- Γ_1 is just the dual polar graph itself.
- Γ_{d+1} is the disjointness-graph: two generators are adjacent iff they are disjoint.

Convention

The number of m-spaces in a projective space $PG(n, q)$ is $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q$,
where $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is the Gaussian coefficient: $\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=1}^{b} \frac{q^{a+1-i}-1}{q^i-1}$.

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Parameters of the dual polar graph

- The valency is $k = q^{\epsilon+1} \begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q$.
- $b_i = q^{i+\epsilon+1} \begin{bmatrix} d+1-i \\ 1 \end{bmatrix}_q$ and $c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q$.
- The $d+2$ eigenvalues are: $q^{\epsilon+1} \begin{bmatrix} d-r \\ 1 \end{bmatrix}_q - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q$ with $-1 \leq r \leq d$.
- The graphs Γ_i have valencies $q^{i(i+1+2\epsilon)/2} \begin{bmatrix} d+1 \\ i \end{bmatrix}_q$.

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We will generalize this for generators in $H(2n+1, q^2)$!

Cliques

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Theorem

If R is any relation of an association scheme $\mathcal{R}$ on Ω, with valency k and $\lambda < 0$, then for any clique S of R: $|S| \leq 1 - k/\lambda$.

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If R corresponds with a strongly regular graph on Ω, this is known as Hoffman's bound.

Proof

The *inner distribution vector* $\mathbf{a} := (\mathbf{a}_0, \ldots, \mathbf{a}_D)$ of $X \subseteq \Omega$ $(X \neq \emptyset)$:

$$\mathbf{a}_i = \frac{1}{|X|}|\{(X \times X) \cap R_i\}|, \quad \text{for all } i \in \{0, \ldots, D\}.$$

Hence $\mathbf{a}_i$ is the average number of $x' \in X$ with $(x, x') \in R_i$, for all $x \in X$.

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If S is a clique of relation R_i: $\mathbf{a} = (1, 0, \ldots, |S| - 1, \ldots, 0)$.

- Theorem: if **a** is inner distribution vector, and P matrix of eigenvalues of association scheme, then all entries of $\mathbf{a}P^{-1}$ are non-negative (Delsarte, 1973).

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- Hence $|S| \leq 1 - k/\lambda_j$ for every negative eigenvalue λ_j of relation R_i.

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If R is a symmetric relation on Ω, a set $S \subseteq \Omega$ is a *clique* of R if $(x, y) \in R, \forall x, y \in S$.

Theorem

If R is any relation of an association scheme $\mathcal{R}$ on Ω, with valency k and $\lambda < 0$, then for any clique S of R: $|S| \leq 1 - k/\lambda$.

How we will use this to study partial spreads:

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We must find eigenvalues λ of the disjointness-relation!

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- Eigenvalues dual polar graph: $q^{\epsilon+1} \begin{bmatrix} d-r \\ 1 \end{bmatrix}_q - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q, -1 \leq r \leq d$.
- For every eigenvalue θ of a distance-regular graph Γ, there is a sequence of eigenvalues v_i of R_i, with $v_0 = 1, v_1 = \theta$, and $\theta v_i = c_{i+1} v_{i+1} + (k - b_i - c_i) v_i + b_{i-1} v_{i-1}, \forall i \in \{1, \ldots, D-1\}$.

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- $\implies$ The disjointness-relation Γ_{d+1} has eigenvalue $(-1)^{d+1}q^{d(d+1)/2}$.

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- Valency $k = (q^2)^{(d+1)(d+2+2\epsilon)/2} = q^{(2n+1)^2}$.
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Hence the upper bound $q^{2n+1}+1$ in $H(4n+1, q^2)$ is tight.

Association scheme on m-spaces

- Ω: the set of m-spaces in $\mathcal{P}$.
- Relations $R_{s,k}$: $(\pi_1, \pi_2) \in R_{s,k} \iff dim(\pi_1 \cap \pi_2) = s$ and $index(\langle \pi_1, \pi_2 \rangle) = k$ (i.e. the maximal dimension of subspaces of $\mathcal{P}$ in $\langle \pi_1, \pi_2 \rangle$ is k).

Example

Ω=Set of lines in $H(5, q^2)$:

- $(l, l') \in R_{1,1} \iff l = l'$,
- $(l, l') \in R_{0,2} \iff l \cap l' \neq \emptyset, \langle l, l' \rangle$ is a plane of $H(5, q^2)$,
- $(l, l') \in R_{0,1} \iff l \cap l' \neq \emptyset, \langle l, l' \rangle$ is not a plane of $H(5, q^2)$,
- $(l, l') \in R_{-1,2} \iff l \cap l' = \emptyset, \langle l, l' \rangle$ contains a plane of $H(5, q^2)$,
- $(l, l') \in R_{-1,1} \iff l \cap l' = \emptyset, \langle l, l' \rangle$ contains no plane of $H(5, q^2)$.

Oppositeness

Two m-spaces π_1 and π_2 are *opposite* if $(\pi_1, \pi_2) \in R_{-1,m}$:
$\iff \pi_1 \cap \pi_2 = \emptyset$ and $\langle \pi_1, \pi_2 \rangle$ contains no $(m+1)$-spaces of $\mathcal{P}$,
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In particular: partial ovoids = partial 0-systems, partial spreads = partial *d*-systems.

Oppositeness-relation on *m*-spaces

- Valency: $k = q^{(m+1)(2d+1+\epsilon-3m/2)}$.
- Eigenvalues:

$$\lambda = (-1)^{r+i+1} q^{i(i+\epsilon)+(r+1)(r/2+d-i-m)+(m-r)(2d-r/2+1/2+\epsilon-3m/2)}$$

  with $-1 \leq r \leq m, 0 \leq i \leq \min(r+1, d-m)$.
- For $r = 0, i = 0 : \lambda = -q^{d-3m/2+m(2d+1+\epsilon-3m/2)}$.

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   Here $k = q^{d(d+1)/2}, \lambda = -q^{d(d+1)/2}$.
   Upper bound: $1 - k/\lambda = 1 + 1 = 2$.
   Very silly result: two disjoint generators have to be of different type, and there are only 2 types!

Theorem

Consider a-spaces and b-spaces in $PG(n, q)$, with $min(a, n-1-a) \leq min(b, n-1-b)$ (hence with more possible sizes of intersections between b-spaces than between a-spaces).
If S is a set of a-spaces s.t. every b-spaces is incident with λ elements of $S \Longrightarrow S$ is empty or the full set of a-spaces.

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Example

There is no non-trivial set S of lines in $\mathrm{PG}(6, q)$ such that every plane contains a constant number of lines of S.

Thank you for your attention!