

An algebraic approach to partial spreads of hermitian varieties

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Our goal is to study partial spreads in polar spaces.

- There are six types of nonsingular classical polar spaces:

\mathcal{P}	ϵ	d
$Q^-(2n+1, q)$	+1	$n-1$
$H(2n, q^2)$	+1/2	$n-1$
$Q(2n, q)$	0	$n-1$
$W(2n+1, q)$	0	n
$H(2n+1, q^2)$	-1/2	n
$Q^+(2n+1, q)$	-1	n

- Number of points in \mathcal{P} : $(q^{d+1+\epsilon} + 1)(q^{d+1} - 1)/(q - 1)$.
- Generators* in a polar space \mathcal{P} = subspaces of max. projective dimension d .

(Partial) Spreads

- A *partial spread* in a polar space \mathcal{P} is a set of pairwise disjoint generators.
- The size of a partial spread in \mathcal{P} is at most:
$$|\mathcal{P}| / \frac{q^{d+1}-1}{q-1} = q^{d+1+\epsilon} + 1 \text{ (=the ovoid number of } \mathcal{P}\text{)}.$$
- A *spread* in \mathcal{P} is a partition of \mathcal{P} into generators, or hence a partial spread of size $q^{d+1+\epsilon} + 1$.
- A partial spread is *maximal* if it cannot be extended to a larger partial spread.

Questions

- When does a polar space \mathcal{P} have a spread?
- If it doesn't, what size can its partial spreads still have?

Results on spreads in non-Hermitian polar spaces

\mathcal{P}	Existence of spreads
$Q^-(2n+1, q)$	q even: YES
$Q(2n, q)$	q odd, n even: NO; q even: YES
$W(2n+1, q)$	YES
$Q^+(2n+1, q)$	n even: NO; q even, n odd: YES

Hermitian variety $H(2n + 1, q^2)$

Our main focus is on the Hermitian variety $H(2n + 1, q^2)$:
Subspaces in $PG(2n + 1, q^2)$, all points (X_0, \dots, X_{2n+1}) of which
satisfy: $X_0^{q+1} + \dots + X_{2n+1}^{q+1} = 0$.

Properties

- $d = n$ and $\epsilon = -1/2$.
- Ovoid number: $q^{2n+1} + 1$.

What is already known on partial spreads in $H(2n + 1, q^2)$?

- J. A. Thas (1990): $H(2n + 1, q^2)$ has no spreads, or hence no partial spreads of size $q^{2n+1} + 1$.
- Aguglia, Cossidente & Ebert (2001): construction of partial spreads of size $q^{n+1} + 1$ in $H(2n + 1, q^2)$.
- De Beule & Metsch (2007): the maximum size of a partial spread in $H(5, q^2)$ is $q^3 + 1$.

New result

We will give a generalisation of the last result...
by use of algebraic combinatorics!

Definition

D -class *association scheme* on finite set Ω :

Set of symmetric relations $\mathcal{R} = \{R_0, R_1, \dots, R_D\}$ on Ω s.t.:

- (i) R_0 is the identity relation,
- (ii) \mathcal{R} is a partition of Ω^2 ,
- (iii) there are *intersection numbers* p_{ij}^k such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k .

Every relation R_i is regular, with valency p_{ij}^0 .

Example 1

Hamming scheme $H(n, q)$ on a set Γ of size q :

- $\Omega: \Gamma^n$, i.e. all words of length n over Γ ,
- $(\alpha, \beta) \in R_i \iff \alpha$ and β differ in i positions, $0 \leq i \leq n$.

Example 2: Three association schemes derived from $H(5, q^2)$

- Ω =Set of points of $H(5, q^2)$:

- $(p, p') \in R_0 \iff p = p'$,
- $(p, p') \in R_1 \iff \langle p, p' \rangle$ is a line of $H(5, q^2)$,
- $(p, p') \in R_2 \iff \langle p, p' \rangle$ is not a line of $H(5, q^2)$.

- Ω =Set of lines of $H(5, q^2)$:

- $(l, l') \in R_0 \iff l = l'$,
- $(l, l') \in R_1 \iff l \cap l' \neq \emptyset, \langle l, l' \rangle$ is a plane of $H(5, q^2)$,
- $(l, l') \in R_2 \iff l \cap l' \neq \emptyset, \langle l, l' \rangle$ is not a plane of $H(5, q^2)$;
- $(l, l') \in R_3 \iff l \cap l' = \emptyset, \langle l, l' \rangle$ contains a plane of $H(5, q^2)$,
- $(l, l') \in R_4 \iff l \cap l' = \emptyset, \langle l, l' \rangle$ contains no plane of $H(5, q^2)$.

- Ω =Set of planes of $H(5, q^2)$:

- $(\pi, \pi') \in R_0 \iff \pi = \pi'$,
- $(\pi, \pi') \in R_1 \iff \pi \cap \pi'$ is a line,
- $(\pi, \pi') \in R_2 \iff \pi \cap \pi'$ is a point,
- $(\pi, \pi') \in R_3 \iff \pi \cap \pi' = \emptyset$.

Real algebras over finite sets

- $\mathbb{R}\Omega$: the real algebra over $\Omega = \{\omega_1, \dots, \omega_\Omega\} : \{\sum_{i=1}^{|\Omega|} c_i \omega_i \mid c_i \in \mathbb{R}\}$.
- The set $\{\omega_1, \dots, \omega_{|\Omega|}\}$ is a basis of $\mathbb{R}\Omega$.
- Every relation $R \subseteq (\Omega \times \Omega)$ defines an endomorphism on $\mathbb{R}\Omega$:

$$R(\omega) := \sum_{(\omega, \omega') \in R} \omega', \quad \forall \omega \in \Omega.$$

- Matrix representation A of relation R with respect to basis $\{\omega_1, \dots, \omega_{|\Omega|}\}$ of $\mathbb{R}\Omega$:

$$A_{ij} = \begin{cases} 1 & \text{if } (\omega_i, \omega_j) \in R, \\ 0 & \text{if } (\omega_i, \omega_j) \notin R. \end{cases}$$

R is a symmetric relation $\iff A$ is a symmetric matrix.

The Bose-Mesner algebra of an association scheme

$$\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \\ a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

- Every relation R_i in a D -class association scheme $\mathcal{R} = \{R_0, \dots, R_D\}$ defines a symmetric endomorphism on $\mathbb{R}\Omega$.
- From Definition: If $(x, y) \in R_k$, there are p_{ij}^k vertices $z \in \Omega$ s.t. $(x, z) \in R_i$ and $(z, y) \in R_j$.
 Hence: $R_i(R_j(v)) = R_j(R_i(v)) = \sum_{k=0}^D p_{ij}^k R_k(v), \forall v \in \mathbb{R}\Omega$.
- The Bose-Mesner algebra of \mathcal{R} is the algebra generated by R_0, \dots, R_D .

Orthogonal decomposition of Bose-Mesner algebra

If \mathcal{R} is a D -class association scheme on Ω , then there is a unique orthogonal decomposition $\mathbb{R}\Omega = V_0 \perp \dots \perp V_D$, such that all V_i are eigenspaces of every relation R_j .

Eigenvalues

The *matrix of eigenvalues* is a $(D + 1) \times (D + 1)$ -matrix P , with P_{ij} the eigenvalue of R_j for V_i .

Example: Matrix of eigenvalues of Hamming scheme $H(3, q)$

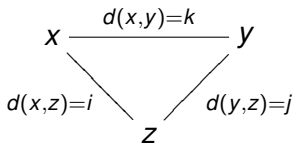
$\Omega = \Gamma^3$ with $|\Gamma| = q$,

$(\alpha, \beta) \in R_i \iff \alpha$ and β differ in i positions, with $0 \leq i \leq 3$.

	R_0	R_1	R_2	R_3
V_0	1	$3(q-1)$	$3(q-1)^2$	$(q-1)^3$
V_1	1	$2q-3$	q^2-4q+3	$-(q-1)^2$
V_2	1	$q-3$	$-2q+3$	$q-1$
V_3	1	-3	3	-1

Distance-regular graphs

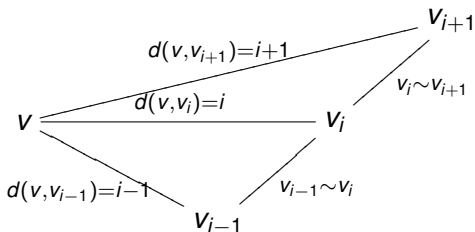
- Consider a connected graph Γ on V with diameter D .
- Define graphs Γ_i on V for every $i \in \{0, \dots, D\}$:
 $v_1 \sim v_2 \iff d(v_1, v_2) = i$, and let R_i denote the corresponding symmetric relation on V .
- Γ is distance-regular if $\{R_0, \dots, R_D\}$ is an association scheme on V , or hence if:
 There are constants p_{ij}^k s.t. $\forall x, y$ with $d(x, y) = k$ there are p_{ij}^k vertices z with $d(x, z) = i$ and $d(z, y) = j$.



Weakening of axioms

A connected graph Γ is also distance-regular if there are constants k , b_i and c_i s.t.:

- Γ is regular with valency k .
- If $d(v, v_i) = i$, there are c_i neighbours v_{i-1} of v_i with $d(v, v_{i-1}) = i - 1$, $\forall i \in \{1, \dots, D\}$.
- If $d(v, v_i) = i$, there are b_i neighbours v_{i+1} of v_i with $d(v, v_{i+1}) = i + 1$, $\forall i \in \{0, \dots, D - 1\}$.



Sequences of eigenvalues

With every eigenvalue θ of Γ corresponds a sequence of eigenvalues v_0, v_1, \dots, v_D of $\Gamma_0, \Gamma_1, \dots, \Gamma_D$, with $v_0 = 1, v_1 = \theta$ and $\theta v_i = c_{i+1} v_{i+1} + (k - b_i - c_i) v_i + b_{i-1} v_{i-1}, \quad \forall i \in \{1, \dots, D-1\}$.

Example: Distance-regular graph Γ on planes in $H(5, q^2)$

$V =$ Set of planes in $H(5, q^2)$.

Two planes are connected in Γ if they meet in a line.

- $\Gamma_0 : d(\pi, \pi') = 0 \iff \pi = \pi'$,
- $\Gamma_1 : d(\pi, \pi') = 1 \iff \pi \cap \pi'$ is a line,
- $\Gamma_2 : d(\pi, \pi') = 2 \iff \pi \cap \pi'$ is a point,
- $\Gamma_3 : d(\pi, \pi') = 3 \iff \pi \cap \pi' = \emptyset$.

Definition

Dual polar graph Γ on generators (or d -spaces) of polar space \mathcal{P} : two generators π_1, π_2 are adjacent iff $\dim(\pi_1 \cap \pi_2) = d - 1$.

Properties

- $d(\pi_1, \pi_2) = i$ if and only if $\pi_1 \cap \pi_2$ is a $(d - i)$ -space.
- Γ is a distance-regular graph with diameter $d + 1$.
- In Γ_0 , every generator is connected only to itself.
- Γ_1 is just the dual polar graph itself.
- Γ_{d+1} is the disjointness-graph: two generators are adjacent iff they are disjoint.

Convention

The number of m -spaces in a projective space $\text{PG}(n, q)$ is $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q$,

where $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is the Gaussian coefficient: $\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=1}^b \frac{q^{a+1-i}-1}{q^i-1}$.

Parameters of the dual polar graph

- The valency is $k = q^{\epsilon+1} \begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q$.
- $b_i = q^{i+\epsilon+1} \begin{bmatrix} d+1-i \\ 1 \end{bmatrix}_q$ and $c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q$.
- The $d+2$ eigenvalues are: $q^{\epsilon+1} \begin{bmatrix} d-r \\ 1 \end{bmatrix}_q - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q$ with $-1 \leq r \leq d$.
- The graphs Γ_i have valencies $q^{i(i+1+2\epsilon)/2} \begin{bmatrix} d+1 \\ i \end{bmatrix}_q$.

The distance-regular graph on planes of $H(5, q^2)$ was an example of a dual polar graph.

We will generalize this for generators in $H(2n + 1, q^2)$!

Cliques

If R is a symmetric relation on Ω , a set $S \subseteq \Omega$ is a *clique* of R if $(x, y) \in R, \forall x \neq y \in S$.

Theorem

If R is any relation of an association scheme \mathcal{R} on Ω , with valency k and $\lambda < 0$, then for any clique S of R : $|S| \leq 1 - k/\lambda$.

If R corresponds with a strongly regular graph on Ω , this is known as Hoffman's bound.

Proof

The *inner distribution vector* $\mathbf{a} := (\mathbf{a}_0, \dots, \mathbf{a}_D)$ of $X \subseteq \Omega$ ($X \neq \emptyset$):

$$\mathbf{a}_i = \frac{1}{|X|} |\{(X \times X) \cap R_i\}|, \quad \text{for all } i \in \{0, \dots, D\}.$$

Hence \mathbf{a}_i is the average number of $x' \in X$ with $(x, x') \in R_i$, for all $x \in X$.

If S is a clique of relation R_i : $\mathbf{a} = (1, 0, \dots, |S| - 1, \dots, 0)$.

- Theorem: if \mathbf{a} is inner distribution vector, and P matrix of eigenvalues of association scheme, then all entries of $\mathbf{a}P^{-1}$ are non-negative (Delsarte, 1973).
- Identity: $P^{-1} = \frac{\Delta_n^{-1} P^T \Delta_m}{|\Omega|}$, with Δ_n and Δ_m diagonal matrices, with the valencies of relations and dimensions of eigenspaces on the diagonal, respectively.
- Hence all entries of $\mathbf{a}\Delta_n^{-1} P^T$ must also be non-negative.
- If $\mathbf{a} = (1, 0, \dots, |S| - 1, \dots, 0)$, then $(\mathbf{a}\Delta_n^{-1} P^T)_j = 1 + \frac{(|S|-1)\lambda_j}{k}$.
- Hence $|S| \leq 1 - k/\lambda_j$ for every negative eigenvalue λ_j of relation R_j . □

Cliques

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Theorem

If R is any relation of an association scheme \mathcal{R} on Ω , with valency k and $\lambda < 0$, then for any clique S of R : $|S| \leq 1 - k/\lambda$.

How we will use this to study partial spreads:

- Ω : the set of generators or d -spaces in \mathcal{P} ,
- Association scheme \mathcal{R} : induced by dual polar graph of \mathcal{P} ,
- Relation R of \mathcal{R} : the disjointness-relation,
- Cliques of R = partial spreads.

We must find eigenvalues λ of the disjointness-relation!

Reminders

- Eigenvalues dual polar graph: $q^{\epsilon+1} \begin{bmatrix} d-r \\ 1 \end{bmatrix}_q - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q, -1 \leq r \leq d$.
- For every eigenvalue θ of a distance-regular graph Γ , there is a sequence of eigenvalues v_i of R_i , with $v_0 = 1$, $v_1 = \theta$, and $\theta v_i = c_{i+1} v_{i+1} + (k - b_i - c_i) v_i + b_{i-1} v_{i-1}, \forall i \in \{1, \dots, D-1\}$.

Calculations

- Take eigenvalue $\theta = - \begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q$ (so $r = d$) of the dual polar graph.
- Sequence of eigenvalues v_i of Γ_i :
 $v_i = (-1)^i q^{\binom{i}{2}} \begin{bmatrix} d+1 \\ i \end{bmatrix}_q, \forall i \in \{0, \dots, d+1\}$.
- \implies The disjointness-relation Γ_{d+1} has eigenvalue $(-1)^{d+1} q^{d(d+1)/2}$.

Theorem

The size of a partial spread in $H(4n + 1, q^2)$ is at most $q^{2n+1} + 1$.

Proof

In $H(4n + 1, q^2)$, the maximal dimension d is $2n$ and $\epsilon = -1/2$.

Disjointness-relation:

- Valency $k = (q^2)^{(d+1)(d+2+2\epsilon)/2} = q^{(2n+1)^2}$.
- Eigenvalue $\lambda = (-1)^{d+1}(q^2)^{d(d+1)/2} = -q^{2n(2n+1)}$.

If S is a partial spread (or clique): $|S| \leq 1 - k/\lambda = 1 + q^{2n+1}$. □

Tightness

Reminder: Aguglia, Cossidente & Ebert (2001): construction of partial spreads of size $q^{n+1} + 1$ in $H(2n + 1, q^2)$,

Hence the upper bound $q^{2n+1} + 1$ in $H(4n + 1, q^2)$ is tight.

Association scheme on m -spaces

- Ω : the set of m -spaces in \mathcal{P} .
- Relations $R_{s,k}$: $(\pi_1, \pi_2) \in R_{s,k} \iff \dim(\pi_1 \cap \pi_2) = s$ and $\text{index}(\langle \pi_1, \pi_2 \rangle) = k$ (i.e. the maximal dimension of subspaces of \mathcal{P} in $\langle \pi_1, \pi_2 \rangle$ is k).

Example

Ω =Set of lines in $H(5, q^2)$:

- $(l, l') \in R_{1,1} \iff l = l'$,
- $(l, l') \in R_{0,2} \iff l \cap l' \neq \emptyset, \langle l, l' \rangle$ is a plane of $H(5, q^2)$,
- $(l, l') \in R_{0,1} \iff l \cap l' \neq \emptyset, \langle l, l' \rangle$ is not a plane of $H(5, q^2)$,
- $(l, l') \in R_{-1,2} \iff l \cap l' = \emptyset, \langle l, l' \rangle$ contains a plane of $H(5, q^2)$,
- $(l, l') \in R_{-1,1} \iff l \cap l' = \emptyset, \langle l, l' \rangle$ contains no plane of $H(5, q^2)$

Oppositeness

Two m -spaces π_1 and π_2 are *opposite* if $(\pi_1, \pi_2) \in R_{-1,m}$:
 $\iff \pi_1 \cap \pi_2 = \emptyset$ and $\langle \pi_1, \pi_2 \rangle$ contains no $(m+1)$ -spaces of \mathcal{P} ,
 \iff every subspace of \mathcal{P} through π_1 is disjoint from π_2 .
For generators: oppositeness = disjointness!

Partial m -systems

A *partial m -system* is a set S of pairwise opposite m -spaces, i.e. a clique of $R_{-1,m}$.
In particular: partial ovoids = partial 0-systems, partial spreads = partial d -systems.

Oppositeness-relation on m -spaces

- Valency: $k = q^{(m+1)(2d+1+\epsilon-3m/2)}$.
- Eigenvalues:

$$\lambda = (-1)^{r+i+1} q^{i(i+\epsilon)+(r+1)(r/2+d-i-m)+(m-r)(2d-r/2+1/2+\epsilon-3m/2)}$$

with $-1 \leq r \leq m, 0 \leq i \leq \min(r+1, d-m)$.

- For $r = 0, i = 0$: $\lambda = -q^{d-3m/2+m(2d+1+\epsilon-3m/2)}$.

A general upper bound on size of cliques (=partial m -systems)

Taking $r, i = 0$: $|S| \leq 1 - k/\lambda = 1 + q^{d+1+\epsilon}$ =ovoid number of \mathcal{P} .

Old bound: partial m -systems meeting it are m -systems.

In particular: ovoids = 0-systems, spreads = d -systems.

But is λ always minimized for $r, i = 0$?

Yes, except when....

- 1 $\epsilon = -1/2, r = m = d, i = 0$ and d is even, hence in $H(2d + 1, q)$ with d even.

Here $k = q^{(d+1)^2/2}, \lambda = -q^{(d+1)d/2}$.

Upper bound: $1 - k/\lambda = 1 + q^{(d+1)/2}$.

Or: $1 + q^{2n+1}$ in $H(4n + 1, q^2)$.

- 2 $\epsilon = -1, r = m = d, i = 0$ and d is even, hence in $Q^+(2d + 1, q)$ with d even.

Here $k = q^{d(d+1)/2}, \lambda = -q^{d(d+1)/2}$.

Upper bound: $1 - k/\lambda = 1 + 1 = 2$.

Very silly result: two disjoint generators have to be of different type, and there are only 2 types!

Theorem

Consider a -spaces and b -spaces in $\text{PG}(n, q)$, with $\min(a, n - 1 - a) \leq \min(b, n - 1 - b)$ (hence with more possible sizes of intersections between b -spaces than between a -spaces).
If S is a set of a -spaces s.t. every b -space is incident with λ elements of $S \implies S$ is empty or the full set of a -spaces.

Example

There is no non-trivial set S of lines in $\text{PG}(6, q)$ such that every plane contains a constant number of lines of S .

Thank you for your attention!