Incidence geometry from an algebraic graph theory point of view

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Outline

- Incidence geometries: examples, problems,...
- Graph theory
- Algebraic techniques in graph theory
  and some applications in incidence geometry.

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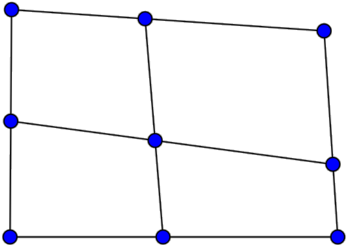
- Many of these incidence structures are *spherical buildings* (Tits) (related to Coxeter groups)

Example: grids (rank=2)



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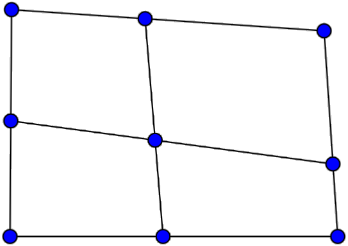

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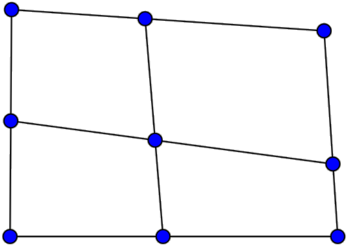

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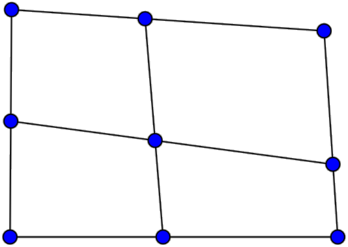

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Properties:

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- 2 distinct points are either on 1 common line (collinear) or none.
- If $p \not\mathrm{I}\, \ell$ then there is a unique $q \,\mathrm{I}\, \ell$ collinear with p.

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A *partial spread of lines*: a subset of the 2-dimensional subspaces in P_2, all pairwise intersecting trivially
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What is the maximum size for partial spreads of lines?

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- A subspace is *isotropic* if f vanishes on it.
- Let P_i be set of isotropic i-dimensional subspaces.
- Incidence I is just symmetrized inclusion.
- This incidence structure is a *classical polar space*, with rank maximal dimension of isotropic subspaces.

Graphs

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- a finite set Ω: the *vertices*
- a set E of pairs of vertices: the *edges*.

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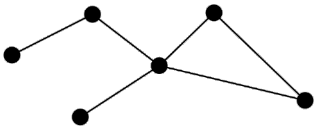

Regular graphs

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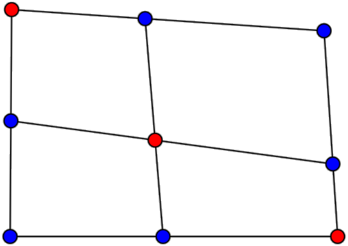
Cliques and cocliques in graphs

- Clique: subset of vertices, every two elements adjacent.
- Coclique: subset of vertices, no two elements adjacent.

Collinearity graph of the grid

Vertices: 9 points. Adjacency: collinearity.



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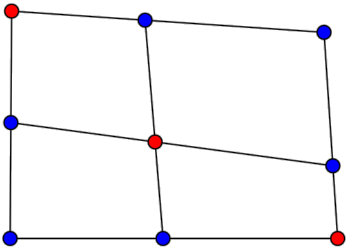

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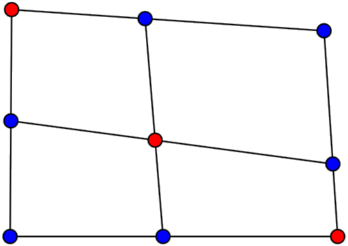

- Collinearity graph is regular with valency 4.
- The red points are a coclique of size 3 (and we can't do better).

Graphs from projective space $\mathrm{PG}(n,q)$

Grassmann graph $J_q(n+1,2)$ in $V(n+1,q)$:

- 2-dimensional subspaces as vertices
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Finding partial spreads of lines in $\mathrm{PG}(n, q)$=
Finding cocliques in $J_q(n+1, 2)$.

Graphs from polar spaces

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Dual polar graph:

- vertices: isotropic subspaces of maximal dimension d
- x and y are adjacent if $x \cap y$ is $(d-1)$-dimensional.

Consider graph Γ with vertex set Ω:

- Adjacency matrix A: $(|\Omega| \times |\Omega|)$-matrix:

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are adjacent} \\ 0 & \text{if } x \text{ and } y \text{ are not adjacent} \end{cases} .$$

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- If Γ is regular with valency k then $A(1,1,\ldots,1)^T = k(1,1,\ldots,1)^T$, so k is eigenvalue (and largest eigenvalue).

Characteristic vectors

For any subset of vertices S,
the *characteristic vector* $\chi_S \in \mathbb{R}^\Omega$ is $(0, 1, 0, 1, 1, \ldots)^T$
with $(\chi_S)_\omega = 1$ if $\omega \in S$ and $(\chi_S)_\omega = 0$ if $\omega \notin S$.

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Just $\langle E_i\chi_S, E_i\chi_S\rangle \geq 0$ might already be powerful!

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- This yields new (tight) bound for one specific type of polar space!

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- Erdős-Ko-Rado-sets = cocliques in Γ!

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- Nice things happen in case of equality!

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- Joint work with V. Pepe and L. Storme: using properties in case of equality to characterize Erdős-Ko-Rado-sets of maximum size.
- Characterization was possible for almost all polar spaces,
  easy construction is unique optimal construction in most cases.

Thank you for your attention!

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