Incidence geometry from an algebraic graph theory point of view

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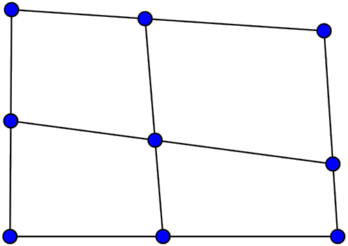
Outline

- Incidence geometries: examples, problems,...
- Graph theory
- Algebraic techniques in graph theory
  and some applications in incidence geometry.

- We study *incidence geometries* (finite or infinite).
- An *incidence geometry of rank n* is a pair $(\{P_1, \ldots, P_n\}, \mathrm{I})$ with n disjoint sets of objects $P_1, \ldots, P_n$,
  and a symmetric incidence relation I between objects of $\neq$ type.
- Objects are often referred to as *points*, *lines*, *planes*,....
- Plenty of examples: projective spaces, polar spaces,
  generalized polygons, unitals, inversive planes,....
- Many of these incidence structures are *spherical buildings* (Tits)
  (related to Coxeter groups)

Example: grids (rank=2)



P_1: 9 points, P_2: 6 lines.
Properties:

- Every point is on 2 lines.
- Every line contains 3 points.
- 2 distinct points are either on 1 common line (collinear) or none.
- If $p \not\mathrm{I}\, \ell$ then there is a unique $q \,\mathrm{I}\, \ell$ collinear with p.

Example: projective space $\mathrm{PG}(n, q)$

- We build a rank n incidence geometry from vector space $V(n+1, q)$.
- Objects in P_i are i-dimensional subspaces, with $i \in \{1, \ldots, n\}$.
- Incidence I is just symmetrized inclusion.

Example of problem in $\mathrm{PG}(n, q)$

A *partial spread of lines*: a subset of the 2-dimensional subspaces in P_2, all pairwise intersecting trivially
(i.e. not incident with a common 1-dimensional subspace or *point*).
What is the maximum size for partial spreads of lines?

Example: (classical) polar spaces

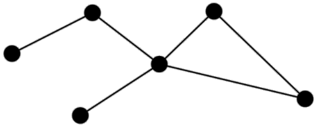
- Consider $V(n,q)$ and a non-singular quadratic, symmetric, alternating of hermitian form f.
- A subspace is *isotropic* if f vanishes on it.
- Let P_i be set of isotropic i-dimensional subspaces.
- Incidence I is just symmetrized inclusion.
- This incidence structure is a *classical polar space*, with rank maximal dimension of isotropic subspaces.

Graphs

A *graph* Γ consists of:

- a finite set Ω: the *vertices*
- a set E of pairs of vertices: the *edges*.

x and y are *adjacent* or *neigbours* if $\{x, y\}$ is an edge of Γ.



Regular graphs

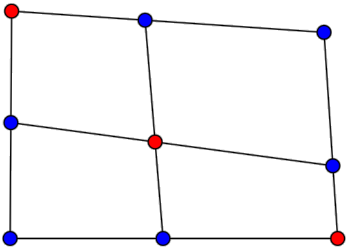
A graph is *regular with valency k* if each vertex has exactly k neighbours.

Cliques and cocliques in graphs

- Clique: subset of vertices, every two elements adjacent.
- Coclique: subset of vertices, no two elements adjacent.

Collinearity graph of the grid

Vertices: 9 points. Adjacency: collinearity.



- Collinearity graph is regular with valency 4.
- The red points are a coclique of size 3 (and we can't do better).

Graphs from projective space $\mathrm{PG}(n, q)$

Grassmann graph $J_q(n+1, 2)$ in $V(n+1, q)$:

- 2-dimensional subspaces as vertices
- x and y are adjacent if $dim(x \cap y) = 1$.

Finding partial spreads of lines in $\mathrm{PG}(n, q)$=
Finding cocliques in $J_q(n+1, 2)$.

Graphs from polar spaces

(Remember: (classical) polar space consists of isotropic subspaces in $V(n, q)$ with respect to form f.)

Dual polar graph:

- vertices: isotropic subspaces of maximal dimension d
- x and y are adjacent if $x \cap y$ is $(d-1)$-dimensional.

Consider graph Γ with vertex set Ω:

- Adjacency matrix A: $(|\Omega| \times |\Omega|)$-matrix:

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are adjacent} \\ 0 & \text{if } x \text{ and } y \text{ are not adjacent} \end{cases}.$$

- *Eigenvalues of* Γ= the eigenvalues of A.
- A is real-symmetric $\Longrightarrow$ all eigenvalues are real.
- If Γ is regular with valency k then $A(1,1,\ldots,1)^T = k(1,1,\ldots,1)^T$, so k is eigenvalue (and largest eigenvalue).

Characteristic vectors

For any subset of vertices S,
the *characteristic vector* $\chi_S \in \mathbb{R}^{\Omega}$ is $(0,1,0,1,1,\ldots)^T$
with $(\chi_S)_\omega = 1$ if $\omega \in S$ and $(\chi_S)_\omega = 0$ if $\omega \notin S$.

Orthogonal decomposition into eigenspaces

We can write:

$$I = E_1 + \ldots + E_d,$$

with E_i orthogonal projections (idempotents)
onto eigenspaces of adjacency matrix A.
So:

$$\chi_S = E_1\chi_S + \ldots + E_d\chi_S.$$

Just $\langle E_i\chi_S, E_i\chi_S\rangle \geq 0$ might already be powerful!

Partial spread problem in polar spaces

- Remember: classical polar space of rank d consists of isotropic subspaces in $V(n, q)$ w.r.t form f,
  with d maximal dimension of isotropic subspaces.
- Partial spread: subset of d-dimensional spaces,
  pairwise intersecting trivially.
- Consider the (regular) graph Γ with d-dimensional spaces as vertices, and 2 vertices adjacent when intersecting trivially.
- Partial spreads = cliques in Γ.
- Delsarte's *linear programming bound* yields that for any clique S:

$$|S| \leq 1 - k/\lambda_{min}$$

  with k the valency of Γ and λ_{min} minimal eigenvalue.
- This yields new (tight) bound for one specific type of polar space!

The converse problem

- Consider d-dimensional isotropic subspaces in (classical) polar space of rank d again.
- An Erdős-Ko-Rado-set is a set of isotropic d-dimensional subspaces, no two intersecting trivially.
- Consider the (regular) graph Γ with d-dimensional spaces as vertices, and 2 vertices adjacent when intersecting trivially.
- Erdős-Ko-Rado-sets = cocliques in Γ!

General theorem on cocliques (Hoffman)

- Suppose Γ is regular with valency k, vertex set Ω and adjacency matrix A.
- S is coclique (so no 2 elements of S adjacent) $\iff (\chi_S)^T A \chi_S = 0$.
- If λ_{min} is minimal eigenvalue of Γ then $A - \lambda_{min} I$ is positive semidefinite: $v^T(A - \lambda_{min} I)v \geq 0, \forall v \in \mathbb{R}^\Omega$.
- Working out

$$\left(\chi_S - \frac{|S|}{|\Omega|}(1,1,\ldots,1)^T\right)^T (A - \lambda_{min} I)\left(\chi_S - \frac{|S|}{|\Omega|}(1,1,\ldots,1)^T\right) \geq 0$$

yields:

$$|S| \leq \frac{|\Omega|}{1 - k/\lambda_{min}}.$$

- Nice things happen in case of equality!

Characterizing Erdős-Ko-Rado-families of maximum size

- An Erdős-Ko-Rado-set is a set of isotropic d-dimensional subspaces, no two intersecting trivially.
- Easy (and big) Erdős-Ko-Rado-set:
  all isotropic d-subspaces through fixed 1-dimensional subspace.
- Stanton (1980) used Hoffman's bound to compute bounds for size.
- Joint work with V. Pepe and L. Storme: using properties in case of equality to characterize Erdős-Ko-Rado-sets of maximum size.
- Characterization was possible for almost all polar spaces,
  easy construction is unique optimal construction in most cases.

Thank you for your attention!

(Slides (and more) on `http://cage.ugent.be/~fvanhove`)