Eigenvalue techniques applied to polar spaces

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Seminar on Incidence Geometry
October 16, 2009

Outline

- Introduction
- Algebraic background
- Geometric applications

Theorem (Thas, 1976)

If S is a set of mutually disjoint planes in $Q(6, q)$ or $W(5, q)$, then $|S| \leq q^3 + 1$. If equality occurs, all planes $\notin S$ meet a unique element of S in a line. (In Thas' terminology: spreads are perfect 1-codes.)

Proof.

Let M be the set of planes meeting a (necessarily unique) element of S in a line and R the rest.
There are $(q+1)(q^2+1)(q^3+1)$ planes in total, $q+1$ planes through a line and q^2+q+1 lines in a plane:

$$(q+1)(q^2+1)(q^3+1) = |S|+|M|+|R| \geq |S|+|M| = |S|+|S|(q^2+q+1)q,$$

so $q^3+1 \geq |S|$ with $|R| = 0$ if $|S| = q^3+1$. □

Can we obtain similar results with other techniques?

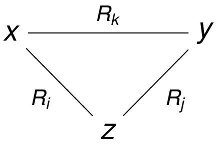
Definition of association scheme

A D-class association scheme is a pair $(\Omega, \{R_0, \dots, R_D\})$ with Ω a set and $R_0, \dots, R_D$ symmetric relations on Ω s.t.:

(i) R_0 is the identity relation,

(ii) $\{R_0, \dots, R_D\}$ is a partition of Ω^2,

(iii) there are *intersection numbers* p_{ij}^k such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

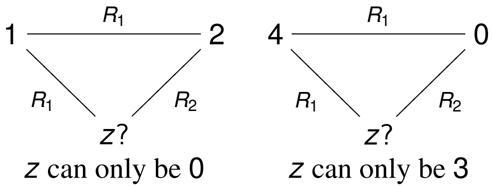


Every relation R_i is thus regular, with valency $k_i = p_{ii}^0$.

Example: the Paley scheme $P_5 = (\Omega, \{R_0, R_1, R_2\})$

- $\Omega = \mathbb{F}_5 = \{0, 1, 2, 3, 4\}$.
- We define 3 relations R_0, R_1, R_2:
  - R_0 is the identity relation: e.g. $(2, 2) \in R_0$,
  - $(a, b) \in R_1$ if $a - b$ is 1 or 4 (and hence square): e.g. $(2, 3) \in R_1$,
  - $(a, b) \in R_2$ if $a - b$ is 2 or 3 (and hence non-square): e.g. $(1, 4) \in R_2$.

Intersection number p^1_{12}:



One should check all such cases... to conclude: $p^1_{12} = 1$.

Definition of Schur idempotents A_i

Consider an association scheme $(\Omega, \{R_0, \ldots, R_D\})$ and order the elements of Ω: $\omega_1, \ldots, \omega_{|\Omega|}$. For each relation R_i, define the $(|\Omega| \times |\Omega|)$-matrix A_i over $\mathbb{R}$:

$$\begin{cases} (A_i)_{rs} & = & 1 \text{ if } (\omega_r, \omega_s) \in R_i, \\ (A_i)_{rs} & = & 0 \text{ if } (\omega_r, \omega_s) \notin R_i. \end{cases}$$

Properties

- A_0 is the identity matrix.
- $A_0 + \ldots + A_D$ is all-one matrix.
- They are $D + 1$ linearly independent matrices.
- A_i is symmetric.
- $A_i A_j = \sum_k p_{ij}^k A_k = \sum_k p_{ij}^k (A_k)^t = (A_i A_j)^t = A_j^t A_i^t = A_j A_i$.

Example: Paley scheme $P_5 = (\{0,1,2,3,4\},\{R_0,R_1,R_2\})$

$$A_0 = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array},$$

$$A_1 = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}, A_2 = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

Note that A_0, A_1 and A_2 are symmetric and add up to the all-one matrix!

Definition Bose-Mesner algebra

Consider an association scheme $(\Omega, \{R_0, \ldots, R_D\})$.
Bose-Mesner algebra: $(D+1)$-space spanned by matrices $A_0, \ldots, A_D$.

Properties of Bose-Mesner algebra

- Closed under multiplication as $A_iA_j = \sum_k p_{ij}^k A_k, \forall i, j$.
- Commutative as $A_iA_j = A_jA_i, \forall i, j$.

Problem with basis $\{A_0, \ldots, A_D\}$.

Calculating the multiplication $(\alpha_0 A_0 + \ldots + \alpha_D A_D)(\beta_0 A_0 + \ldots + \beta_D A_D)$ is very very exhausting!

Minimal idempotents

There is a unique basis $\{E_0, \ldots, E_D\}$ of the Bose-Mesner algebra such that E_i is idempotent (so $E_i^2 = E_i$) and $E_iE_j = 0$ if $i \neq j$.
$E_0, \ldots E_D$ are the minimal idempotents.

Properties of minimal idempotents

- E_i is an orthogonal projection, and projection onto $(1, \ldots, 1)$ is one of them (usually E_0).
- $\mathbb{R}^{|\Omega|}$ orthogonally decomposes as $\mathrm{Im}(E_0) \perp \ldots \mathrm{Im}(E_D)$.
- $E_0 + \ldots + E_D = I = A_0$.
- $(\lambda_0 E_0 + \ldots + \lambda_D E_D)(\mu_0 E_0 + \ldots + \mu_D E_D)$ is simply $(\lambda_0 \mu_0 E_0 + \ldots + \lambda_D \mu_D E_D)$.
- If $A_i = \lambda_0 E_0 + \ldots + \lambda_D E_D$ and $v = vE_j$ (so $v \in \mathrm{Im}(E_i)$) then $vA_i = (vE_j)(\lambda_0 E_0 + \ldots + \lambda_D E_D) = v(\lambda_j E_j) = \lambda_j v$.
  So the coefficients $\lambda_0, \ldots, \lambda_D$ are precisely the eigenvalues of A_i!

Matrix of eigenvalues

The matrix of eigenvalues P is the $(D+1) \times (D+1)$-matrix containing all eigenvalues:

$$\begin{pmatrix} 1 & \dots & P_{0i} = k_i & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & P_{Di} & \dots \end{pmatrix},$$

with $A_i = P_{0i}E_0 + \ldots + P_{Di}E_D$.

Properties

- As E_0 is orthogonal projection onto all-one vector, the first row consists of the valencies.
- As $A_0 = I$, all its eigenvalues are 1 and hence the first column is all-one.
- As $A_0 + \ldots + A_D$ is all-one matrix, it is $|\Omega|E_0$, and hence the sum of the columns is $(|\Omega|, 0, \ldots, 0)^t$.

Matrix of eigenvalues of Paley scheme P_5

$P_5 = (\mathbb{F}_5, \{R_0, R_1, R_2\})$ with R_0, R_1, R_2 the identity, square-difference- and non-square-difference-relation.

$$P = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} \begin{matrix} A_0 & A_1 & A_2 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 2 \\ 1 & \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \\ 1 & \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \end{pmatrix} \end{array} \begin{matrix} E_0 \\ E_1 \\ E_2 \end{matrix}$$

so: $A_0 = E_0 + E_1 + E_2$,
$A_1 = 2E_0 + (\frac{-1-\sqrt{5}}{2})E_1 + (\frac{-1+\sqrt{5}}{2})E_2$, $A_2 = 2E_0 + (\frac{-1+\sqrt{5}}{2})E_1 + (\frac{-1-\sqrt{5}}{2})E_2$.

- the first row contains valencies of R_0, R_1, R_2,
- the first column is all-one,
- the sum of the columns is $(5, 0, 0)^t$!

How to express minimal idempotents

The columns of matrix of eigenvalues give coefficients for A_i as combination of $E_0, \ldots, E_D$.
But conversely, how to express E_j as a combination of $A_0, \ldots, A_D$?
Inverting the matrix P is quite hard.....
Theorem: if for some E_j, every R_i has valency k_i and eigenvalue λ_i, then E_j is up to a positive scalar: $\frac{\lambda_0}{k_0} A_0 + \ldots + \frac{\lambda_D}{k_D} A_D$.

Example: expressions for Paley scheme P_5

$$P = \begin{array}{c} \begin{matrix} A_0 & A_1 & A_2 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 2 \\ 1 & \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \\ 1 & \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \end{pmatrix} \end{array} \begin{matrix} E_0 \\ E_1 \\ E_2 \end{matrix}$$

$$\begin{cases} A_0 = E_0 + E_1 + E_2 \\ A_1 = 2E_0 + (\frac{-1-\sqrt{5}}{2})E_1 + (\frac{-1+\sqrt{5}}{2})E_2 \\ A_2 = 2E_0 + (\frac{-1+\sqrt{5}}{2})E_1 + (\frac{-1-\sqrt{5}}{2})E_2 \end{cases}$$

$$\begin{cases} E_0 \sim \frac{1}{1}A_0 + \frac{2}{2}A_1 + \frac{2}{2}A_2 \\ E_1 \sim \frac{1}{1}A_0 + \frac{(-1-\sqrt{5})/2}{2}A_1 + \frac{(-1+\sqrt{5})/2}{2}A_2 \\ E_2 \sim \frac{1}{1}A_0 + \frac{(-1+\sqrt{5})/2}{2}A_1 + \frac{(-1-\sqrt{5})/2}{2}A_2 \end{cases}$$

Suppose $(\Omega, \{R_0, \ldots, R_D\})$ is an association scheme, and (non-empty) $S \subseteq \Omega$. Choose a numbering $\omega_1, \ldots, \omega_{|\Omega|}$ for Ω.

Characteristic vector of a subset

The characteristic vector χ_S of S has $(\chi_S)_i = 1$ if $\omega_i \in S$ and $(\chi_S)_i = 0$ if $\omega_i \notin S$. So $\chi_S = (0, 1, 0, 1, 1, \ldots)$.

The vector $v_{\omega,S}$

For every $\omega \in \Omega$, we define the $(D+1)$-vector $v_{\omega,S}$ as:

$$v_{\omega,S} = [|R_0 \cap (\{\omega\} \times S)|, \ldots, |R_D \cap (\{\omega\} \times S)|],$$

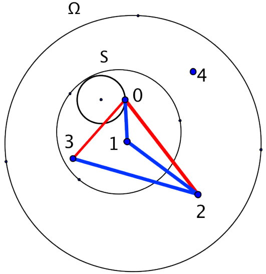
So $(v_{\omega,S})_i = |R_i \cap (\{\omega\} \times S)| = \chi_S A_i \chi^t_{\{\omega\}}$.

Inner distribution of S

The inner distribution $\mathbf{a}$ is the average of all $v_{\omega,S}$ with $\omega \in S$.
So $\mathbf{a}_i = (\chi_S A_i \chi^t_S)/|S|$.

The Paley Scheme $P_5 = (\{0,1,2,3,4\}, \{R_0, R_1, R_2\})$

Consider $S = \{0,1,3\}$. Characteristic vector of S: $\chi_S = (1,1,0,1,0)$.



$v_{0,S} = [1, 1, 1]$ (see drawing)
$v_{1,S} = [1, 1, 1]$
$v_{2,S} = [0, 2, 1]$ (see drawing)
$v_{3,S} = [1, 0, 2]$
$v_{4,S} = [0, 2, 1]$

Inner distribution of S: $\mathbf{a} = (v_{0,S} + v_{1,S} + v_{3,S})/3 = [1, \frac{2}{3}, \frac{4}{3}]$.

We can orthogonally decompose χ_S as $\chi_S E_0 + \ldots + \chi_S E_D$.
What if some components are zero?

Theorem (Delsarte, 1973)

Let $(\Omega, \{R_0, \ldots, R_D\})$ be an association scheme with $S \subseteq \Omega$. For any minimal idempotent E_j, let λ_i be the eigenvalue of A_i, and let k_i be the valency of R_i. The inner distribution $\mathbf{a}$ of S satisfies:

$$\frac{\lambda_0}{k_0}\mathbf{a}_0 + \ldots + \frac{\lambda_D}{k_D}\mathbf{a}_D \geq 0,$$

with equality if and only if $\chi_S E_j = 0$.

Proof

$0 \leq (\chi_S E_j)(\chi_S E_j)^t = \chi_S E_j E_j \chi_S^t = \chi_S E_j \chi_S^t \sim$
$\chi_S(\frac{\lambda_0}{k_0}A_0 + \ldots + \frac{\lambda_D}{k_D}A_D)\chi_S^t \sim \frac{\lambda_0}{k_0}\mathbf{a}_0 + \ldots + \frac{\lambda_D}{k_D}\mathbf{a}_D$. □

If $S \subset \Omega$ satisfies $\chi_S E_j = 0$, then $\chi_S E_j \chi^t_{\{\omega\}} = 0, \forall \omega \in \Omega$.
This implies:
$0 = \chi_S(\frac{\lambda_0}{k_0} A_0 + \ldots + \frac{\lambda_D}{k_D} A_D)\chi^t_{\{\omega\}} = \frac{\lambda_0}{k_0}(v_{\omega,S})_0 + \ldots + \frac{\lambda_D}{k_D}(v_{\omega,S})_D, \forall \omega \in \Omega$.

Main idea

The more minimal idempotents E_j for which $\chi_S E_j = 0$, the nicer the subset S!!

Example: the Paley scheme P_q

Suppose $q \equiv 1 \bmod 4$ is a prime power. Let Ω be $\mathbb{F}_q$, with R_0 the identity relation, $(a, b) \in R_1$ if $a - b$ is a non-zero square, and $(a, b) \in R_2$ if $a - b$ is non-square.
Now $(\Omega, \{R_0, R_1, R_2\})$ is an association scheme with matrix of eigenvalues P:

$$\begin{pmatrix} 1 & (q-1)/2 & (q-1)/2 \\ 1 & (-1-\sqrt{q})/2 & (-1+\sqrt{q})/2 \\ 1 & (-1+\sqrt{q})/2 & (-1-\sqrt{q})/2 \end{pmatrix}$$

Theorem

If $q \equiv 1 \bmod 4$ and every difference in $S \subset \mathbb{F}_q$ is square, then $|S| \leq \sqrt{q}$. If $|S| = \sqrt{q}$, then $\forall a \in \mathbb{F}_q \backslash S$, the number of $s \in S$ with $a - s$ square is $(\sqrt{q} - 1)/2$.

Proof

$$\begin{pmatrix} 1 & (q-1)/2 & (q-1)/2 \\ 1 & (-1-\sqrt{q})/2 & (-1+\sqrt{q})/2 \\ 1 & (-1+\sqrt{q})/2 & (-1-\sqrt{q})/2 \end{pmatrix}$$

$\forall s \in S : v_{s,S} = (1, |S| - 1, 0) \Longrightarrow$ Inner distribution : $\mathbf{a} = (1, |S| - 1, 0)$.
2nd row:$0 \leq \frac{1}{1}\mathbf{a}_0 + \frac{(-1-\sqrt{q})/2}{(q-1)/2}\mathbf{a}_1 + \frac{(-1+\sqrt{q})/2}{(q-1)/2}\mathbf{a}_2 = 1 + \frac{|S|-1}{1-\sqrt{q}}$, with equality iff $\chi_S E_1 = 0$, so $|S| \leq \sqrt{q}$ with equality iff $\chi_S E_1 = 0$.
If $|S| = \sqrt{q}$ and $a \in \mathbb{F}_q \backslash S$ then: $v_{a,S} = (0, x, \sqrt{q} - x)$.
$0 = \frac{1}{1}(v_{a,S})_0 + \frac{(-1-\sqrt{q})/2}{(q-1)/2}(v_{a,S})_1 + \frac{(-1+\sqrt{q})/2}{(q-1)/2}(v_{a,S})_2$
$\Longrightarrow x = (\sqrt{q} - 1)/2$. □

Polar spaces

- Polar space $\mathcal{P}$: incidence structure with 0-spaces (points), 1-spaces(lines),. . .,$(d-1)$-spaces,d-spaces (generators)
- (Finite) classical polar spaces:

$\mathcal{P}$	pts. on line	d-spaces through $(d-1)$-space	e
$Q^+(2d+1,q)$	$(q)+1$	$(q)^0+1$	0
$H(2d+1,q^2)$	$(q^2)+1$	$(q^2)^{\frac{1}{2}}+1$	$1/2$
$W(2d+1,q)$	$(q)+1$	$(q)^1+1$	1
$Q(2d+2,q)$	$(q)+1$	$(q)^1+1$	1
$H(2d+2,q^2)$	$(q^2)+1$	$(q^2)^{\frac{3}{2}}+1$	$3/2$
$Q^-(2d+3,q)$	$(q)+1$	$(q)^2+1$	2

- $q+1$ points on line$\rightarrow$ $(q^{d+e}+1)(q^{d+1}-1)/(q-1)$ points in $\mathcal{P}$

Association scheme $(\Omega, \{R_0, \ldots, R_{d+1}\})$ on generators of polar space

- Ω: generators or d-spaces of polar space
- $(\pi, \pi') \in R_i$ iff $\pi \cap \pi'$ is $(d - i)$-dimensional

In particular: R_0 is identity relation, and $(\pi, \pi') \in R_{d+1}$ iff $\pi \cap \pi' = \emptyset$.

Example: planes or 2-spaces in $Q^-(7,q)$

- Four relations:
  $R_0 =$ identity, R_1=meet in line, $R_2 =$ meet in point, $R_3 =$ disjoint.
- Matrix of eigenvalues:

$$\begin{pmatrix} 1 & q^2\left(q^2+q+1\right) & q^5\left(q^2+q+1\right) & q^9 \\ 1 & q^3+q^2-1 & \left(q^3-q-1\right)q^2 & -q^5 \\ 1 & q^2-q-1 & -\left(q^2+q-1\right)q & q^3 \\ 1 & -q^2-q-1 & q\left(q^2+q+1\right) & -q^3 \end{pmatrix}$$

(there is a natural ordering for the rows here)
Not really elegant... but consider last row/first row!

$$E_3 \sim A_0 - \frac{1}{q^2}A_1 + \frac{1}{q^4}A_2 - \frac{1}{q^6}A_3$$

- In general, if $\mathcal{P}$ has q^e+1 d-spaces through every $(d-1)$-space:

$$E_{d+1} \sim A_0 + \left(\frac{-1}{q^e}\right) A_1 + \ldots + \left(\frac{-1}{q^e}\right)^{d+1} A_{d+1}.$$

- Different cases:

$$\begin{aligned}
Q^+(2d+1,q): E_{d+1} &\sim A_0 - A_1 + \ldots + (-1)^{d+1} A_{d+1} \\
H(2d+1,q^2): E_{d+1} &\sim A_0 - \frac{1}{(q^2)^{1/2}} A_1 + \ldots + \left(\frac{-1}{(q^2)^{1/2}}\right)^{d+1} A_{d+1} \\
W(2d+1,q)/Q(2d+1,q): E_{d+1} &\sim A_0 - \frac{1}{q} A_1 + \ldots + \left(\frac{-1}{q}\right)^{d+1} A_{d+1} \\
H(2d+2,q^2): E_{d+1} &\sim A_0 - \frac{1}{(q^2)^{3/2}} A_1 + \ldots + \left(\frac{-1}{(q^2)^{3/2}}\right)^{d+1} A_{d+1} \\
Q^-(2d+3,q): E_{d+1} &\sim A_0 - \frac{1}{q^2} A_1 + \ldots + \left(\frac{-1}{q^2}\right)^{d+1} A_{d+1}
\end{aligned}$$

For every subset $S \subseteq \Omega$ of generators: $\chi_S = \chi_S E_0 + \ldots + \chi_S E_{d+1}$ (with $\chi_S E_0 = |S|/|\Omega|\chi_\Omega$) (use natural ordering for the E_i).
If $S_1, S_2 \subseteq$ then S_1 is a t-design, or S_2 is a t-antidesign, if respectively:

$$\begin{array}{lllllllll} \chi_{S_1} & = & \chi_{S_1}E_0 + & 0 + \ldots + & 0 + & \chi_{S_1}E_{t+1} & + \ldots + & \chi_{S_1}E_{d+1} \\ \chi_{S_2} & = & \chi_{S_2}E_0 + & \chi_{S_2}E_1 + \ldots + & \chi_{S_2}E_t + & 0 & + \ldots + & 0 \end{array}.$$

In particular: S_2 is a d-antidesign if and only if $\chi_{S_2}E_{d+1} = 0$.

Theorem (Delsarte, 1976)

A set S_1 of generators is a t-design if and only if every $(t-1)$-space is incident with exactly λ generators of S_1, for some λ .

t-antidesigns also have a nice property:

Theorem (Delsarte, 1977)

Consider subsets $S_1, S_2 \subset \Omega$ of generators, with S_1 a t-design and S_2 a t-antidesign. Now $|S_1 \cap S_2| = \frac{|S_1||S_2|}{|\Omega|}$.

Proof

We have an orthogonal decomposition:

$$\begin{array}{lclccccccc} \chi_{S_1} & = & \frac{|S_1|}{|\Omega|}\chi_\Omega + & 0 + \ldots + & 0 + & \chi_{S_1}E_{t+1} & + \ldots + & \chi_{S_1}E_{d+1} \\ \chi_{S_2} & = & \frac{|S_2|}{|\Omega|}\chi_\Omega + & \chi_{S_2}E_1 + \ldots + & \chi_{S_2}E_t + & 0 & + \ldots + & 0 \end{array}.$$

So $|S_1 \cap S_2| = \chi_{S_1}\chi_{S_2}^t = \frac{|S_1||S_2|}{|\Omega|^2}\chi_\Omega\chi_\Omega^t = \frac{|S_1||S_2|}{|\Omega|}$. □

We will look for d-antidesigns (so with $\chi_S E_{d+1} = 0$).

Generators of $W(2d+1,q)/Q(2d+2,q)$, embedded in $W(2d+1,q^a)/Q(2d+2,q^a)$

$$W(2d+1,q)/Q(2d+1,q) : E_{d+1} \sim A_0 - \frac{1}{q}A_1 + \ldots + \left(\frac{-1}{q}\right)^{d+1} A_{d+1}$$

$$W(2d+1,q^a)/Q(2d+1,q^a) : E_{d+1} \sim A_0 - \frac{1}{q^a}A_1 + \ldots + \left(\frac{-1}{q^a}\right)^{d+1} A_{d+1}$$

If S is the set of generators of $W(2d+1,q)/Q(2d+2,q)$: inner distribution **a** contains just the valencies

$$\mathbf{a} = \left[1, \ldots, q^{i(i+1)/2} \begin{bmatrix} d+1 \\ i \end{bmatrix}_q, \ldots, q^{(d+1)(d+2)/2}\right]$$

with $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q$ the number of m-spaces in $\mathrm{PG}(n,q)$.

So when is $\chi_S E_{d+1} = 0$? Iff $\mathbf{a}_0 - \frac{1}{q^a}\mathbf{a}_1 + \ldots + (\frac{-1}{q^a})^{d+1}\mathbf{a}_{d+1} = 0$...

So $\chi_S E_{d+1} = 0$ if and only if:

$$\sum_{i=0}^{d+1} (-1)^i q^{-ai} q^{i(i+1)/2} \begin{bmatrix} d+1 \\ i \end{bmatrix}_q = 0.$$

The q-binomial theorem gives us that for any indeterminate z:

$$\sum_{i=0}^{d+1} q^{i(i-1)/2} z^i \begin{bmatrix} d+1 \\ i \end{bmatrix}_q = \prod_{j=1}^{d+1} (1 + q^{j-1} z).$$

With $z = -q^{1-a}$, we see that $E_{d+1}\chi_S = 0 \iff \prod_{j=1}^{d+1}(1 - q^{j-a}) = 0$, which holds if and only if $1 \leq a \leq d+1$.

Application

Let S be the generators of $W(2d+1,q)/Q(2d+2,q)$ in $W(2d+1,q^a)/Q(2d+2,q^a)$ with $1 \leq a \leq d+1$.
For every generator π, we have:
$(v_{\pi,S})_0 - \frac{1}{q^a}(v_{\pi,S})_0 + \ldots + (\frac{-1}{q^a})^{d+1}(v_{\pi,S})_{d+1} = 0.$
If π meets every element of S in at most a point:
$v_{\pi,S} = [0,0,\ldots,0,x,|S|-x]$.
Hence $(\frac{-1}{q^a})^d x + (\frac{-1}{q^a})^{d+1}(|S|-x) = 0 \implies x = \frac{|S|}{q^a+1}$.

(Partial) spreads in a polar space $\mathcal{P}$

$\mathcal{P}$: $q+1$ points on a line, q^e+1 d-spaces through a $(d-1)$-space.

- A *partial spread* S is a set of mutually disjoint generators (or d-spaces) in $\mathcal{P}$.
- Elementary upper bound $|S|$: (#points in $\mathcal{P}$)/(#points in d-space)=$(q^{d+e}+1)$(=*ovoid number*).
- If $|S| = q^{d+e}+1$ (hence if S partitions $\mathcal{P}$) then S is a *spread*.

Algebraic upper bound if d is even

Suppose d is <u>even</u> and S is a partial spread.

$v_{\pi,S} = [1, 0, \ldots, 0, |S|-1], \forall \pi \in S$

$\Longrightarrow$ inner distribution of S:$\mathbf{a} = [1, 0, \ldots, 0, |S|-1]$

$0 \leq \mathbf{a}_0 - \frac{1}{q^e}\mathbf{a}_1 + \ldots + \frac{1}{q^{ed}}\mathbf{a}_d - \frac{1}{q^{e(d+1)}}\mathbf{a}_{d+1} = 1 - \frac{|S|-1}{q^{e(d+1)}}$

So $|S| \leq q^{(d+1)e}+1$ with equality iff $\chi_S E_{d+1} = 0$!

Comparing upper bounds for partial spreads

- So if d is even we have both $q^{d+e}+1$ and $q^{de+e}+1$ as bounds!
- Which bound wins... depends on whether $e<1$, $e=1$ or $e>1$.

$\mathcal{P}$	e	elem. bound	alg. bound
$Q^+(2d+1,q)$	0	q^d+1	$\mathbf{q^0+1=2}$
$H(2d+1,q^2)$	$1/2$	$q^{2d+1}+1$	$\mathbf{q^{d+1}+1}$
$W(2d+1,q)$	1	$\mathbf{q^{d+1}+1}$	$\mathbf{q^{d+1}+1}$
$Q(2d+2,q)$	1	$\mathbf{q^{d+1}+1}$	$\mathbf{q^{d+1}+1}$
$H(2d+2,q^2)$	$3/2$	$\mathbf{q^{2d+3}+1}$	$q^{3d+3}+1$
$Q^-(2d+3,q)$	2	$\mathbf{q^{d+2}+1}$	$q^{2d+2}+1$

$e = 0$: partial spreads S in $Q^+(2d+1, q)$, d even

- $|S| \leq 1 + 1$ with equality iff $\chi_S E_{d+1} = 0$.
- We know that $E_{d+1} \sim A_0 - A_1 + \ldots + A_d - A_{d+1}$.
  So if $|S| = 2$, then for any generator π in $Q^+(2d+1, q)$, we have:
  $(v_{\pi,S})_0 - (v_{\pi,S})_1 + \ldots + (v_{\pi,S})_d - (v_{\pi,S})_{d+1} = 0$.
  Hence $(v_{\pi,S})_a = 1$ and $(v_{\pi,S})_b = 1$ for some odd a and even b, and the rest is zero.

This is normal as there are Latin and Greek d-spaces in $Q^+(2d+1, q)$:

- 2 generators of same type meet in even-dimensional space,
- 2 generators of different type meet in odd-dimensional space.

The 2 elements of S must be of different type, and every generator will meet one in an odd-dimensional space, and the other in an even-dimensional space.

$e = 1/2$: partial spreads S in $H(2d+1, q^2)$, d even

- $|S| \leq q^{d+1} + 1$ with equality iff $\chi_S E_{d+1} = 0$.
- Partial spreads of this size in $H(2d+1, q^2)$ always exist (Aguglia, Cossidente Ebert, 2001).
- We know that $E_{d+1} \sim A_0 - \frac{1}{q}A_1 + \ldots + \frac{1}{q^d}A_d - \frac{1}{q^{d+1}}A_{d+1}$.
  So if $|S| = q^{d+1} + 1$, then for any generator π in $H(2d+1, q^2)$, we have: $(v_{\pi,S})_0 - \frac{1}{q}(v_{\pi,S})_1 + \ldots + \frac{1}{q^d}(v_{\pi,S})_d - \frac{1}{q^{d+1}}(v_{\pi,S})_{d+1} = 0$.

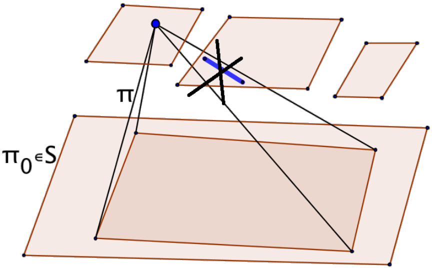
Example: partial spread S of size $2^5 + 1 = 33$ in $H(9, 2^2)$.

For some random 4-spaces (or generators) π, we determine $v_{\pi,S}$.
(So we count the elements of S meeting π in a 4-space, 3-space, 2-space (or plane), 1-space (or line), 0-space (or point) and nothing.)

$$
\begin{array}{ll}
[0,\ 0,\ 0,\ 2,\ 13,\ 18] & -32.0 + 16.0 - 8.0 + 4.2 - 2.13 + 1.18 = 0 \\
[0,\ 0,\ 1,\ 4,\ 12,\ 16] & -32.0 + 16.0 - 8.1 + 4.4 - 2.12 + 1.16 = 0 \\
[1,\ 0,\ 0,\ 0,\ 0,\ 32] & -32.1 + 16.0 - 8.0 + 4.0 - 2.0 + 1.32 = 0 \\
[0,\ 0,\ 0,\ 0,\ 11,\ 22] & -32.0 + 16.0 - 8.0 + 4.0 - 2.11 + 1.22 = 0 \\
[0,\ 1,\ 0,\ 0,\ 16,\ 16] & -32.0 + 16.1 - 8.0 + 4.0 - 2.16 + 1.16 = 0
\end{array}
$$

$(v_{\pi,S})_0 - (v_{\pi,S})_1/2 + (v_{\pi,S})_2/4 - (v_{\pi,S})_3/8 + (v_{\pi,S})_4/16 - (v_{\pi,S})_5/32 = 0$,
or: $-32(v_{\pi,S})_0 + 16(v_{\pi,S})_1 - 8(v_{\pi,S})_2 + 4(v_{\pi,S})_3 - 2(v_{\pi,S})_4 + (v_{\pi,S})_5 = 0$.

Consider a partial spread S of size $q^{d+1}+1$ in $H(2d+1,q^2)$, d even. Suppose π is a generator meeting some $\pi_0 \in S$ in a $(d-1)$-space. Then it meets every other element of S in a point or nothing.



Hence: $v_{\pi,S} = [0,1,0,\dots,0,x,q^{d+1}-x]$.

S: partial spread of size $q^{d+1}+1$ in $H(2d+1,q^2)$, d even.
π : generator meeting some $\pi_0 \in S$ in a $(d-1)$-space.
$v_{\pi,S} = (0,1,0,\dots,0,x,q^{d+1}-x)$.
As $(v_{\pi,S})_0 - \frac{1}{q}(v_{\pi,S})_1 + \dots + \frac{1}{q^d}(v_{\pi,S})_d - \frac{1}{q^{d+1}}(v_{\pi,S})_{d+1} = 0$:

$$-\frac{1}{q} + \frac{x}{q^d} - \frac{q^{d+1}-x}{q^{d+1}} = 0,$$

and hence: $x = q^d \Longrightarrow v_{\pi,S} = [0,1,0,\dots,0,q^d,q^{d+1}-q^d]$.
So we proved:

Property

If S is a partial spread of size $q^{d+1}+1$ in $H(2d+1,q^2)$, d even, then every generator meeting some $\pi_0 \in S$ in a $(d-1)$-space meets exactly q^d elements of S in a point.

(So these partial spreads are 1*-regular codes)*.

S: partial spread of size $q^{d+1}+1$ in $H(2d+1,q^2)$, d even.
π : generator meeting every element of S in at most a point.
$v_{\pi,S} = [0,0,\ldots,0,x,(q^{d+1}+1)-x]$.
As $(v_{\pi,S})_0 - \frac{1}{q}(v_{\pi,S})_1 + \ldots + \frac{1}{q^d}(v_{\pi,S})_d - \frac{1}{q^{d+1}}(v_{\pi,S})_{d+1} = 0$:

$$\frac{x}{q^d} - \frac{(q^{d+1}+1)-x}{q^{d+1}} = 0,$$

and hence: $x = \frac{q^{d+1}+1}{q+1} \implies v_{\pi,S} = [0,0,\ldots,0,\frac{q^{d+1}+1}{q+1},q\frac{q^{d+1}+1}{q+1}]$.
So we proved:

Property

If S is a partial spread of size $q^{d+1}+1$ in $H(2d+1,q^2)$, d even, then every generator meeting **no** element of S in more than a point, meets exactly $\frac{q^{d+1}+1}{q+1}$ elements of S in a point.

Theorem (De Beule & Metsch, 2007)

If S is a partial spread (hence a set of pairwise disjoint planes) in $H(5, q^2)$:

- $|S| \leq q^3 + 1$,
- $|S| = q^3 + 1$ if and only if every free plane (i.e. a plane meeting every element of S in at most a point) meets exactly c elements of S in a point.
  In that case, c must be $\frac{q^3+1}{q+1}$.

(In fact: these partial spreads in $H(5, q^2)$ are *completely regular codes*.)

Theorem

If S is a partial spread of $H(2d+1, q^2)$ with d even, with $|S| = q^{d+1} + 1$,and $T \subset \Omega$ a subset such that each $(d-1)$-space is incident with λ elements of T, then $|S \cap T| = \frac{|S||T|}{|\Omega|}$.

Proof

$\chi_S E_{d+1} = 0$ and hence S is a d-antidesign, while T is a d-design. □

Theorem (Segre, 1965)

If T is a non-trivial set of lines in $H(3, q^2)$, covering every point λ times, then $\lambda = (q+1)/2$.

I don't know any d-designs T in $H(2d+1, q^2)$ (sets of d-spaces covering every $(d-1)$-space λ times)...but $|T|/|\Omega| = 1/2$ must certainly hold (consider residue of $(d-2)$-space).

$e = 1$: partial spreads S in $W(2d+1, q)$ and $Q(2d+2, q)$, d even

- Here, the algebraic and elementary bound are the same: $q^{d+1}+1$.
- So S is a spread (a partition of the full space) iff $\chi_S E_{d+1} = 0$.
- Each spread S is a d-antidesign, so for every d-design T (set of generators covering every $(d-1)$-space λ times): $|S \cap T| = \frac{|S||T|}{|\Omega|}$.
- If S is a spread, then for any generator π in $W(2d+1, q)$ or $Q(2d+2, q)$:
  As $E_{d+1} \sim A_0 - \frac{1}{q}A_1 + \ldots + \frac{1}{q^d}A_d - \frac{1}{q^{d+1}}A_{d+1}$:
  $(v_{\pi,S})_0 - \frac{1}{q}(v_{\pi,S})_1 + \ldots + \frac{1}{q^d}(v_{\pi,S})_d - \frac{1}{q^{d+1}}(v_{\pi,S})_{d+1} = 0$.
  As every point of must π must be covered by exactly one element of S:
  $\frac{q^{d+1}-1}{q-1}(v_{\pi,S})_0 + \frac{q^d-1}{q-1}(v_{\pi,S})_1 + \ldots + 1.(v_{\pi,S})_d + 0.(v_{\pi,S})_{d+1} = \frac{q^{d+1}-1}{q-1}$.

Properties of spread S in $W(2d+1,q)/Q(2d+2,q)$, d even

For every generator or d-space π, we have:

$$(v_{\pi,S})_0 - \frac{1}{q}(v_{\pi,S})_1 + \ldots + \frac{1}{q^d}(v_{\pi,S})_d - \frac{1}{q^{d+1}}(v_{\pi,S})_{d+1} = 0$$

$$\frac{q^{d+1}-1}{q-1}(v_{\pi,S})_0 + \frac{q^d-1}{q-1}(v_{\pi,S})_1 + \ldots + 1.(v_{\pi,S})_d + 0.(v_{\pi,S})_{d+1} = \frac{q^{d+1}-1}{q-1}$$

Let us apply this!

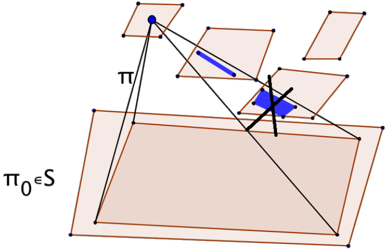
- If $\pi \in S : v_{\pi,S} = [1,0,\ldots,0,q^{d+1}]$.
- If π meets an element of S in a $(d-1)$-space:
  $v_{\pi,S} = [0,1,0,\ldots,0,x,q^{d+1}-x]$
  $\Longrightarrow v_{\pi,S} = [0,1,0,\ldots,0,q^d,q^{d+1}-q^d]$.

Properties of spread S in $W(2d+1,q)/Q(2d+2,q)$, d even

$$(v_{\pi,S})_0 - \frac{1}{q}(v_{\pi,S})_1 + \ldots + \frac{1}{q^d}(v_{\pi,S})_d - \frac{1}{q^{d+1}}(v_{\pi,S})_{d+1} = 0$$

$$\frac{q^{d+1}-1}{q-1}(v_{\pi,S})_0 + \frac{q^d-1}{q-1}(v_{\pi,S})_1 + \ldots + 1.(v_{\pi,S})_d + 0.(v_{\pi,S})_{d+1} = \frac{q^{d+1}-1}{q-1}$$

- If π meets an element of S in a $(d-2)$-space, it meets all others in at most a line: $v_{\pi,S} = [0,0,1,0,\ldots,0,x,y,q^{d+1}-x-y]$.



$\Longrightarrow v_{\pi,S} = [0,0,1,0,\ldots,0,q^{d-2},q^d-q^{d-2},q^{d+1}-q^d]$.
(These spreads are hence 2-*regular codes.)*

Properties of spread S in $W(2d+1,q)/Q(2d+2,q)$, d even

$$(v_{\pi,S})_0 - \frac{1}{q}(v_{\pi,S})_1 + \ldots + \frac{1}{q^d}(v_{\pi,S})_d - \frac{1}{q^{d+1}}(v_{\pi,S})_{d+1} = 0$$

$$\frac{q^{d+1}-1}{q-1}(v_{\pi,S})_0 + \frac{q^d-1}{q-1}(v_{\pi,S})_1 + \ldots + 1.(v_{\pi,S})_d + 0.(v_{\pi,S})_{d+1} = \frac{q^{d+1}-1}{q-1}$$

- If π meets **no element** of S in a plane or more:
  $v_{\pi,S} = [0,0,\ldots,0,x,y,q^{d+1}-x-y]$,
  $\Longrightarrow v_{\pi,S} = [0,0,\ldots,0,\frac{q^d-1}{q^2-1},q^d,q^{d+1}-q^2\frac{q^d-1}{q^2-1}]$.
  So in particular: no generator can meet every element of S in just a point or nothing!

Small case: $W(9, q)$ and $Q(10, q)$.

If S is a spread (hence with $|S| = q^5 + 1$), and π is any 4-space then there are 4 possibilities:

- $\pi \in S$ and is disjoint from all others,
- π meets one element of S in a solid, q^4 in a point and the rest in nothing,
- π meets one element of S in a plane, q^2 in a line, $q^4 - q^2$ in a point and the rest in nothing,
- π meets $q^2 + 1$ elements of S in a line, q^4 in a point and the rest in nothing.

(Spreads are *completely regular codes* in this case.)

Even smaller case: $W(5, q)$ and $Q(6, q)$.

If S is a spread (hence with $|S| = q^3 + 1$), and π is any plane, then there are 2 possibilities:

- $\pi \in S$ and is disjoint from all others,
- π meets one element of S in a line, q^2 in a point and the rest in nothing.

(Spreads are *completely regular codes* and even *perfect 1-codes* in this case.)

Thank you for your attention!