

# Group representations in incidence geometry

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## Outline

- Group representations
- Permutation representations of groups
- Applications in geometry

## Representation of a finite group $G$

A representation is a map  $\rho$  from  $G$  to  $GL(V)$  with

- $V$  a vector space  $V(n, \mathbb{C})$ ,
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$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho(a) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

$$\rho(a^2) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

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- if  $W \subseteq V$  is subspace with  $\rho(g)(W) \subseteq W, \forall g \in G$ :  
 $W$  is *subrepresentation*.

## Homomorphisms between $\rho_1 : G \rightarrow V_1$ and $\rho_2 : G \rightarrow V_2$

A homomorphism is a linear map  $f : V_1 \rightarrow V_2$  with

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Endomorphisms of  $\rho$  form a complex vector space and a ring!

## Irreducible representations

Let  $\rho : G \rightarrow GL(V)$  be a representation.

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## Decomposition of representation $\rho : G \rightarrow GL(V)$

There is a decomposition into irreducible representations:

$$V = (A_1 \oplus \dots \oplus A_a) \oplus (B_1 \oplus \dots \oplus B_b) \oplus \dots$$

with  $(A_1 \dots A_a)$  equivalent,  $(B_1 \dots B_b)$  equivalent,  $\dots$

- unique up to equivalence and ordering
- the subspaces  $(A_1 \oplus \dots \oplus A_a), (B_1 \oplus \dots \oplus B_b), \dots$  are unique!

## Example of decomposition

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$$V = \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

## Dimension of homomorphism-space

- Suppose  $\rho_1 : G \rightarrow GL(V_1)$  and  $\rho_2 : G \rightarrow GL(V_2)$  are representations.
- Remember:  $Hom(V_1, V_2)$  is space of linear maps  $f : V_1 \rightarrow V_2$  with  $f \circ \rho_1(g) = \rho_2(g) \circ f, \forall g \in G$ .

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- We can obtain  $\dim(Hom(V_1, V_2))$  from the decompositions:

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 V_1 = (A_1 \oplus \dots & & \oplus A_{a_1}) \oplus \\
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- $\dim(Hom(V_1, V_2)) = a_1 a_2 + b_1 b_2 + \dots$

## Definition of permutation representation

- Suppose  $G$  acts transitively on (finite) set  $\Omega = \{\omega_1, \omega_2, \dots\}$ .
- Let  $\{\omega_1, \omega_2, \dots\}$  be a basis of  $\mathbb{C}\Omega$ .

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$G = D_8$  and  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$  :

$$\begin{array}{ccc}
 \omega_1 & \text{---} & \omega_2 \\
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 \end{array}$$

$$\rho(\mathbf{e}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \rho((\omega_1\omega_2\omega_3\omega_4)) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \dots$$

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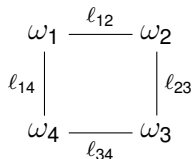
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- These maps  $f_i$  are a basis for  $\text{Hom}(\mathbb{C}\Omega, \mathbb{C}\Omega')$ !

## Example: points and lines of a square

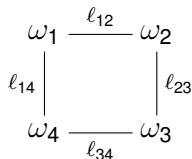
$G = D_8$  acts on  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$  and  $\Omega' = \{\ell_{12}, \ell_{23}, \ell_{34}, \ell_{14}\}$ :



$\Omega \times \Omega'$  splits into orbits  $R_1$  (incident point-line pairs) and  $R_2$  (non-incident point-line pairs).

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$\Omega \times \Omega'$  splits into orbits  $R_1$  (incident point-line pairs) and  $R_2$  (non-incident point-line pairs).

We define corresponding maps  $f_1, f_2$  from  $\mathbb{C}\Omega$  to  $\mathbb{C}\Omega'$ :

- $f_1(\omega_1) = \ell_{12} + \ell_{14}, f_1(\omega_2) = \ell_{12} + \ell_{23}, \dots$
- $f_2(\omega_1) = \ell_{23} + \ell_{34}, f_2(\omega_2) = \ell_{14} + \ell_{34}, \dots$

Now  $\{f_1, f_2\}$  is a basis for  $\text{Hom}(\mathbb{C}\Omega_1, \mathbb{C}\Omega_2)$ .

## Characteristic vector of a subset $S \subseteq \Omega$

*Characteristic vector*  $\chi_S$  is  $(1, 1, 0, 1, 0, \dots)^* \in \mathbb{C}\Omega$ , with  
 $(\chi_S)_{\omega_i} = 1$  if  $\omega_i \in S$ ,  $(\chi_S)_{\omega_i} = 0$  if  $\omega_i \notin S$ .

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## Intersections of two subsets $S_1$ and $S_2$

If  $S_1, S_2 \subseteq \Omega$ , then:

$$\begin{aligned} (\chi_{S_1})^* \chi_{S_2} &= (1, 1, 0, 1, 0, \dots)(0, 1, 1, 1, 0, \dots)^* \\ &= 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 \\ &= |S_1 \cap S_2|. \end{aligned}$$

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Consider any  $S \subseteq \Omega$ . We can decompose accordingly:

$$\chi_S = \frac{|S|}{|\Omega|} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp (v_{A,1} + \dots + v_{A,a}) \perp (v_{B,1} + \dots + v_{B,b}) \perp \dots$$

## Two subsets that are nice with respect to each other

Consider  $S, S' \subseteq \Omega$  with:

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$\forall g \in G$ , we have in this case:

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$$\implies |S \cap (S')^g| = (\chi_S)^*(\chi_{(S')^g}) = |S||S'|/|\Omega|.$$

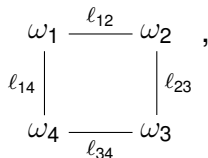
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 $\dim(\text{Hom}(\mathbb{C}\Omega, \mathbb{C}\Omega')) = \text{number of orbits on } \Omega \times \Omega'!$

Example: the square revisited

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- $\dim(\text{Hom}(\mathbb{C}\Omega_1, \mathbb{C}\Omega_1)) = 3$  (=, collinear, opposite points)
- $\dim(\text{Hom}(\mathbb{C}\Omega_2, \mathbb{C}\Omega_2)) = 3$  (=, concurrent and opposite lines)

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 $\dim(\text{Hom}(\mathbb{C}\Omega, \mathbb{C}\Omega')) = \text{number of orbits on } \Omega \times \Omega'!$

Example: the square revisited

$G = D_8$  acts on  $\Omega_1 = \{\omega_1, \omega_2, \omega_3, \omega_4\}$  and  $\Omega_2 = \{\ell_{12}, \ell_{23}, \ell_{34}, \ell_{14}\}$ :

$$\begin{array}{ccc}
 \omega_1 & \xrightarrow{\ell_{12}} & \omega_2 \\
 \ell_{14} \downarrow & & \downarrow \ell_{23} \\
 \omega_4 & \xrightarrow{\ell_{34}} & \omega_3
 \end{array}
 , \mathbb{C}\Omega_1 = \mathbb{C} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp A \perp B, \mathbb{C}\Omega_2 = \mathbb{C} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp A \perp C.$$

- $\dim(\text{Hom}(\mathbb{C}\Omega_1, \mathbb{C}\Omega_1)) = 3$  (=, collinear, opposite points)
- $\dim(\text{Hom}(\mathbb{C}\Omega_2, \mathbb{C}\Omega_2)) = 3$  (=, concurrent and opposite lines)
- $\dim(\text{Hom}(\mathbb{C}\Omega_1, \mathbb{C}\Omega_2)) = 2$  (incidence and non-incidence)

## Our goal

We consider:

- an incidence structure of rank  $n$ : with objects of  $n$  types (points, lines, planes,...)
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We will consider the  $n$  permutation representations on points, lines,...

## Example: the rank $n$ polar space $W(2n - 1, q)$

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$\frac{q^{2n}-1}{q-1}$  points, ...,  $((q + 1) \dots (q^n + 1))$   $n$ -spaces or *generators*.

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Many nice incidence structures are buildings, where  $q = 1$  plays a fundamental role!

Buildings of rank  $n$  are certain incidence structures with  $n$  types (Tits).

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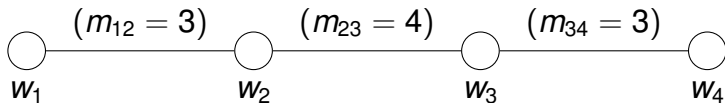
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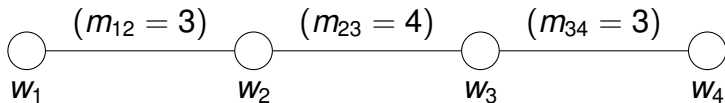
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- The thin building with  $(q_1, \dots, q_n) = (1, \dots, 1)$  is constructed from "Coxeter group" itself, thick building is more complicated but similar.

## Coxeter group $W$ and its thin building



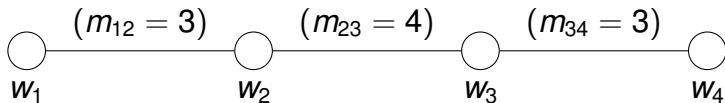
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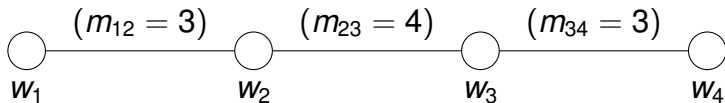
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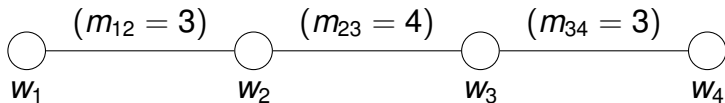
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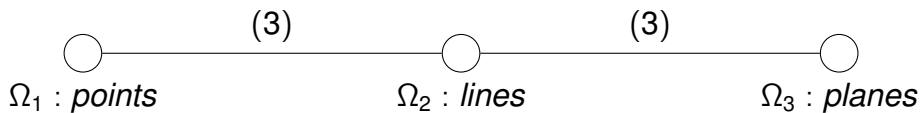


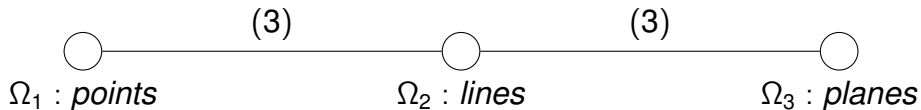
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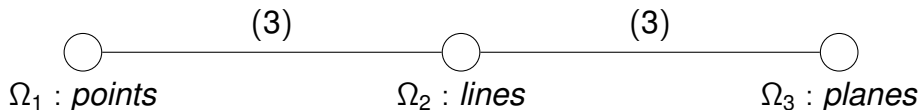


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- $aP_i$  and  $bP_j$  are incident iff  $aP_i \cap bP_j \neq \emptyset$ ,
- decompositions of permutation representation of thick building: in correspondence with that of thin building! (Curtis-Iwahori-Kilmoyer)





We will decompose and write degree for points and lines.



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Thin  $A_3$  building = tetrahedron

■  $\mathbb{C}\Omega_1 = A_1 \perp B_3, \mathbb{C}\Omega_2 = A_1 \perp B_3 \perp C_2.$



Nice sets of lines  $S \subseteq \Omega_2$  in  $PG(3, q)$

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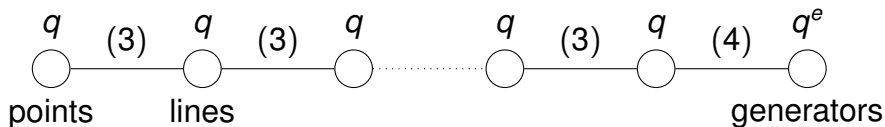
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- $S$  is Cameron-Liebler line class  $\iff \chi_S \in A \perp B \perp \cancel{C}$ .

$B_n$



- $B_n$  is hyperoctahedral group of size  $(2^n n!)$ .
- Thick buildings of type  $B_n$  are the polar spaces.

## Decompositions for (classical) polar spaces

Remember that we just have to look at small group  $B_n$ !

$$\begin{aligned} \mathbb{C}\Omega_1 &= V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \\ \mathbb{C}\Omega_2 &= V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \perp [V^{2,0} \perp V^{2,1} \perp V^{2,2}] \end{aligned}$$

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## Nice sets of points $S$ in the polar space

- $\chi_S \in V^{0,0} \perp V^{1,0} \perp \cancel{V^{1,1}} \iff S$  is *tight set*,
- $\chi_S \in V^{0,0} \perp \cancel{V^{1,0}} \perp V^{1,1} \iff S$  is *k-ovoid*  
 (i.e. every generator meets it in  $k$  points).

The special subspace  $V^{1,0}$

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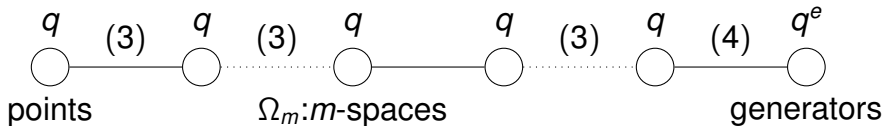
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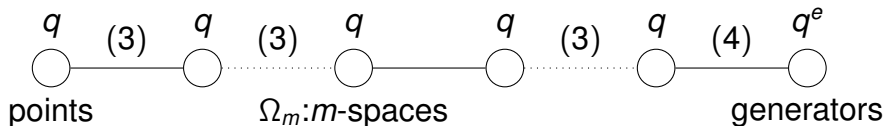
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- For every  $S \subseteq \Omega_m : \langle p^{0,0}(\chi_S), p^{0,0}(\chi_S) \rangle \geq 0$ ,  
 with equality iff  $\chi_S \in (V^{1,0})^\perp$ .

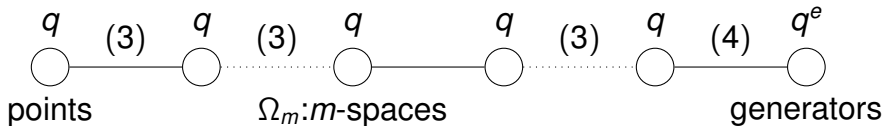


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- In this case:  $\langle p^{1,0}(\chi_S), p^{1,0}(\chi_S) \rangle \geq 0 \iff |S| \leq q^{n-1} \cdot q^e + 1$ .





## Alternative proof for result on $m$ -systems (Shult-Thas)

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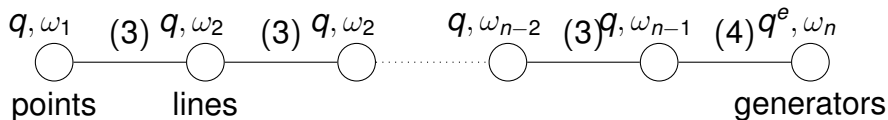
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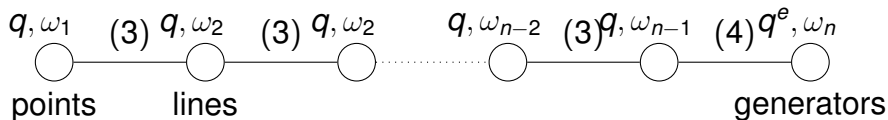
- $m$ -system  $\mathcal{S}$ :  $\chi_S \in V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \perp \dots$
- incidence between  $m$ -spaces and points yields homomorphism  $f : \mathbb{C}\Omega_m \rightarrow \mathbb{C}\Omega_1$ .
- As  $f$  is homomorphism:  $f$  maps subrepr. into corresponding equivalent subrepr.!
- Point-set  $\tilde{\mathcal{S}}$  of points in some  $\pi \in \mathcal{S}$  is given by:  $\chi_{\tilde{\mathcal{S}}} = f(\chi_S)$ .
- $\chi_{\tilde{\mathcal{S}}} \in V^{0,0} \perp [V^{1,0} \perp V^{1,1}]$ .
- Hence  $\tilde{\mathcal{S}}$  is a  $k$ -ovoid  
 (i.e. every generator meets  $\tilde{\mathcal{S}}$  in  $k$  points).



Another interesting subspace (exclusively) for generators

$$\mathbb{C}\Omega_n = V^{0,0} \perp [V^{1,0}] \perp [V^{2,0}] \perp \dots \perp [V^{n-2,0}] \perp [V^{n-1,0}] \perp [V^{n,0}]$$





## Another interesting subspace (exclusively) for generators

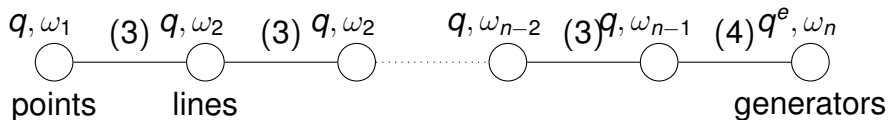
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- $V^{n,0}$  corresponds with 1-dim representation  $\rho$  of  $B_n$ , with:

$$\rho(\omega_1) = \dots = \rho(\omega_{n-1}) = 1, \rho(\omega_n) = -1.$$

- For partial  $n$ -system  $S$ ,  $n$  odd, projection  $p^{n,0} \rightarrow V^{n,0}$  yields:

$$\langle p^{n,0}(\chi_S), p^{n,0}(\chi_S) \rangle \geq 0 \iff |S| \leq 1 - \frac{q^{n^2-n}(q^e)^n}{q^{n^2-n}(-1)^n}.$$



## Another interesting subspace (exclusively) for generators

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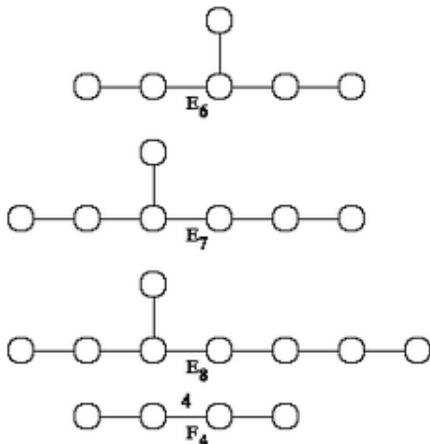
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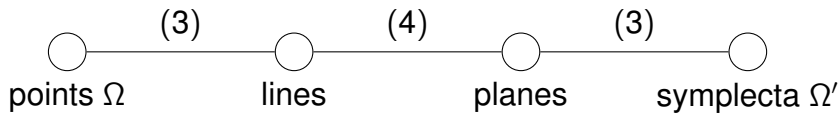
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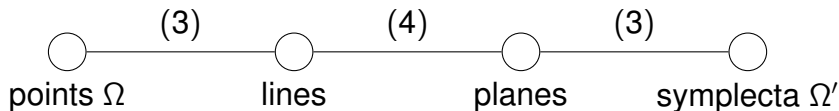
- $\implies$  max. size of partial spread in  $H(4n+1, q^2)$  is  $q^{2n+1} + 1$ .

## What about other finite Coxeter groups?



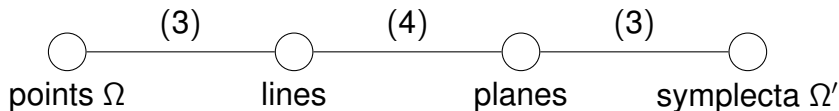


$F_4$



$F_4$

- $|F_4| = 1152 = 2^7 \cdot 3^2$ .
- Thin: 24 points ( $\Omega$ ), 96 lines, 96 planes, 24 symplecta ( $\Omega'$ ).

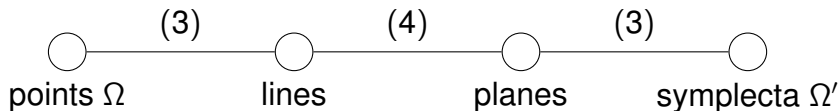


$F_4$

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- Thin: 24 points ( $\Omega$ ), 96 lines, 96 planes, 24 symplecta ( $\Omega'$ ).
- Decompositions of  $\mathbb{C}\Omega$  and  $\mathbb{C}\Omega'$  in thin case with degrees:

$$\mathbb{C}\Omega = J_1 \perp A_4 \perp B_9 \perp C_8 \perp D_2$$

$$\mathbb{C}\Omega' = J_1 \perp A_4 \perp B_9 \perp C'_8 \perp D'_2.$$



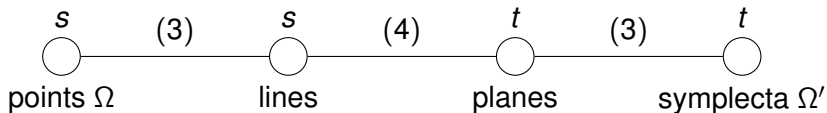
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Only the first 3 irreducible representations are equivalent!

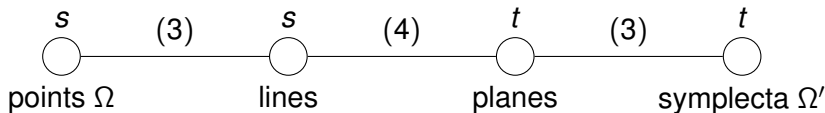


## Thick building of type $F_4$

- $(s, t) = (q, 1), (1, q), (q, q), (q, q^2)$  or  $(q^2, q)$ ,
- number of points  $|\Omega|$  and of symplecta  $|\Omega'|$ :

$$|\Omega| = (s^2 + s + 1)(s^2t + 1)(s^2t^2 + 1)(s^3t^3 + 1),$$

$$|\Omega'| = (t^2 + t + 1)(t^2s + 1)(t^2s^2 + 1)(t^3s^3 + 1).$$



## Thick building of type $F_4$

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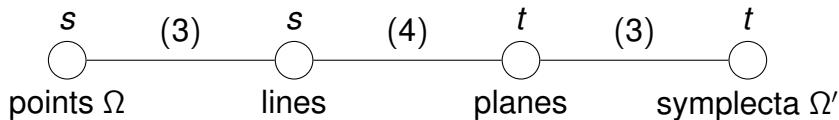
$$|\Omega| = (s^2 + s + 1)(s^2t + 1)(s^2t^2 + 1)(s^3t^3 + 1),$$

$$|\Omega'| = (t^2 + t + 1)(t^2s + 1)(t^2s^2 + 1)(t^3s^3 + 1).$$

- Decompositions of permutation repr. on points and symplecta:

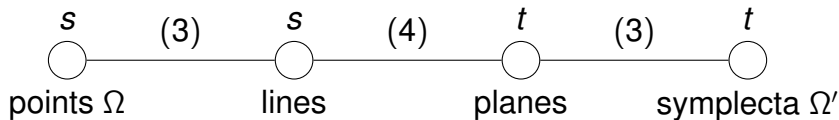
$$\mathbb{C}\Omega = J \perp A \perp B \perp C \perp D$$

$$\mathbb{C}\Omega' = J \perp A \perp B \perp C' \perp D'$$



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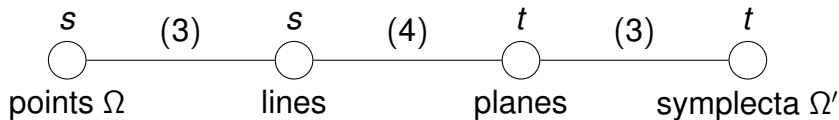


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## Nice subsets of points in $\Omega$

- Set of points  $S$  in 1 symplecton  $\pi$ :  
 consider incidence map  $f \in \text{Hom}(\mathbb{C}\Omega', \mathbb{C}\Omega)$ :

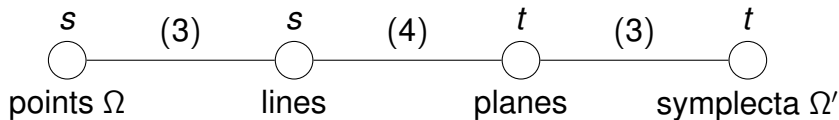


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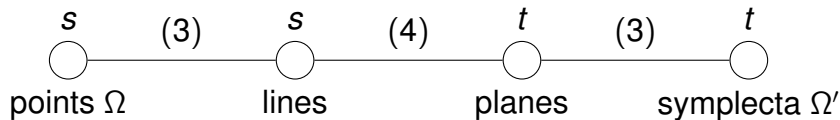


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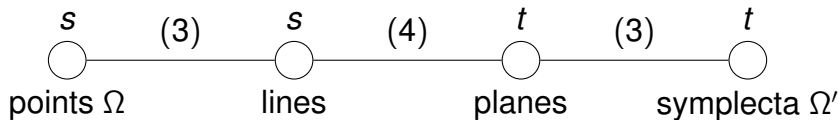
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- Hence if  $\chi_T \in J \perp \cancel{A} \perp \cancel{B} \perp C \perp D$ ,  
then every symplecton meets  $T$  in  $|S||T|/|\Omega|$  points.



### The special subspace $A$

$$\mathbb{C}\Omega = J \perp A \perp B \perp C \perp D.$$

$A$  corresponds with the *reflection representation* of degree 4 of Coxeter group  $F_4$ .



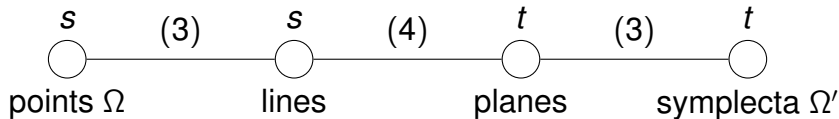
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- $S \subset \Omega$  is *partial ovoid* if no 2 points  $a, b \in S$  are on common lines with a third point.



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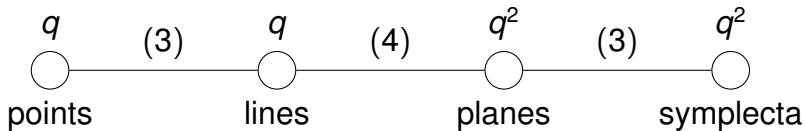
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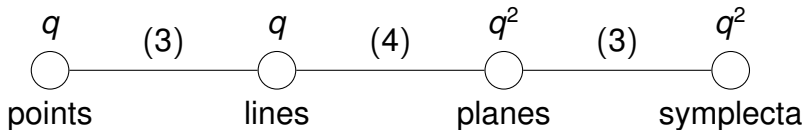
## Obtaining bounds

- $S \subset \Omega$  is *partial ovoid* if no 2 points  $a, b \in S$  are on common lines with a third point.
- Projection  $p : \mathbb{C}\Omega \rightarrow A$  yields for partial ovoids  $S$ :  
 $|S| \leq 1 + s^3 t^3$ , with equality iff  $\chi_S \in J \perp A \perp B \perp C \perp D$ .

An application for  $(s, t) = (q, q^2)$ !



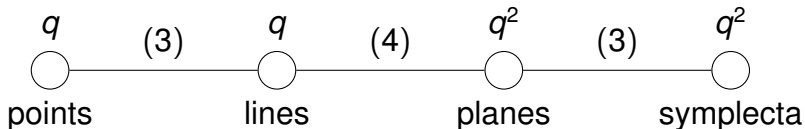
An application for  $(s, t) = (q, q^2)$ !



Suppose

- $T$ : point set of embedded  $F_4$ -building with  $(s', t') = (q, q)$ :  
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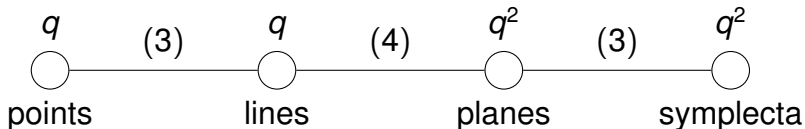
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An application for  $(s, t) = (q, q^2)$ !



Suppose

- $T$ : point set of embedded  $F_4$ -building with  $(s', t') = (q, q)$ :  
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- $S$ : partial ovoid with  $|S| = 1 + s^3 t^3 = 1 + q^9$ :  
 $\chi_S \in J \perp A \perp B \perp C \perp D$ .

$$\implies |S \cap T| = (\chi_S)^* \chi_T = |S||T|/|\Omega| = q^3 + 1.$$

# Thank you for your attention!

(Slides will appear on <http://cage.ugent.be/~fvanhove> )