Group representations in incidence geometry

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Outline

- Group representations
- Permutation representations of groups
- Applications in geometry

Representation of a finite group G

A representation is a map ρ from G to $GL(V)$ with

- V a vector space $V(n, \mathbb{C})$,
- $GL(V)$ the group of invertible linear transformations of V,
- $\rho(g_2g_1) = \rho(g_2)\rho(g_1)$ (so group morphism).

Example

$G = C_3 = \{e, a, a^2\}$ and $V = V(2, \mathbb{C})$.

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho(a) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

$$\rho(a^2) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Representation of a finite group *G* (repeat)

A representation is a map ρ from G to $GL(V)$ with

- V a vector space $V(n, \mathbb{C})$,
- $GL(V)$ the group of invertible linear transformations of V,
- $\rho(g_2 g_1) = \rho(g_2)\rho(g_1)$ (so group morphism).

Properties and terminology

- $\rho(e) = \rho(e.e) = \rho(e).\rho(e) \Longrightarrow \rho(e)$ is identity,
- *degree of representation* ρ is $\dim(V)$,
- if $W \subseteq V$ is subspace with $\rho(g)(W) \subseteq W, \forall g \in G$: W is *subrepresentation*.

Homomorphisms between $\rho_1 : G \to V_1$ and $\rho_2 : G \to V_2$

A homomorphism is a linear map $f : V_1 \to V_2$ with $f \circ \rho_1(g) = \rho_2(g) \circ f, \forall g \in G$.

$$\begin{tikzcd}
V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "f"'] & V_1 \arrow[d, "f"] \\
V_2 \arrow[r, "\rho_2(g)"'] & V_2
\end{tikzcd}$$

The homomorphisms form a complex vector space $Hom(V_1, V_2)$.
ρ_1 and ρ_2 are *equivalent* if a bijective homomorphism exists.

Endomorphisms of a representation ρ

Endomorphisms of ρ: homomorphisms from ρ to ρ.
Endomorphisms of ρ form a complex vector space and a ring!

Irreducible representations

Let $\rho : G \to GL(V)$ be a representation.

- If $W \subset V$ is subspace with $\rho(g)(W) \subseteq W, \forall g \in G$: W is *subrepresentation*.
- V is *irreducible* if $\{0\}$, V are the only subrepresentations.

Decomposition of representation $\rho : G \to GL(V)$

There is a decomposition into irreducible representations:

$$V = (A_1 \oplus \ldots \oplus A_a) \oplus (B_1 \oplus \ldots \oplus B_b) \oplus \ldots$$

with $(A_1 \ldots A_a)$ equivalent, $(B_1 \ldots B_b)$ equivalent, $\ldots$.

- unique up to equivalence and ordering
- the subspaces $(A_1 \oplus \ldots \oplus A_a), (B_1 \oplus \ldots \oplus B_b), \ldots$ are unique!

Example of decomposition

$G = C_2 = \{e, a\}$ and $V = V(2, \mathbb{C})$:

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$V = \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Dimension of homomorphism-space

- Suppose $\rho_1 : G \rightarrow GL(V_1)$ and $\rho_2 : G \rightarrow GL(V_2)$ are representations.
- Remember: $Hom(V_1, V_2)$ is space of linear maps $f : V_1 \rightarrow V_2$ with $f \circ \rho_1(g) = \rho_2(g) \circ f, \forall g \in G$.
- We can obtain $\dim(Hom(V_1, V_2))$ from the decompositions:

$$\begin{tikzcd}
V_1 = (A_1 \oplus \dots \arrow[d] \arrow[dr] & \oplus A_{a_1})\oplus \arrow[dl] \arrow[d] & (B_1 \oplus \dots \arrow[d] \arrow[dr] & \oplus B_{b_1}) \oplus \dots \arrow[dl] \arrow[d] \\
V_2 = (A_2 \oplus \dots & \oplus A_{a_2})\oplus & (B_1 \oplus \dots & \oplus B_{b_2}) \oplus \dots
\end{tikzcd}$$

Each arrow between equivalent irreducibles yields homomorphism!

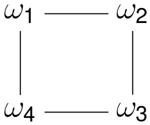
- $\dim(Hom(V_1, V_2)) = a_1 a_2 + b_1 b_2 + \dots$.

Definition of permutation representation

- Suppose G acts transitively on (finite) set $\Omega = \{\omega_1, \omega_2, \ldots\}$.
- Let $\{\omega_1, \omega_2, \ldots\}$ be a basis of $\mathbb{C}\Omega$.
- We define $\rho : G \to GL(\mathbb{C}\Omega)$ by $\rho(g)(\omega_i) = (\omega_i)^g$.

Example

$G = D_8$ and $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$:



.

$$\rho(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \rho((\omega_1\omega_2\omega_3\omega_4)) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ldots$$

Homomorphisms between permutation representations

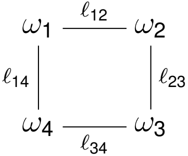
- Suppose G acts transitively on Ω and Ω'.
- $\Omega \times \Omega'$ splits into orbits under G: $R_1 \cup R_2 \cup \ldots$.
- For any R_i, define $f_i : \mathbb{C}\Omega \to \mathbb{C}\Omega'$ by:

$$f_i(\omega_r) = \sum_{(\omega_r, \omega'_s) \in R_i} \omega'_s.$$

- These maps f_i are a basis for $Hom(\mathbb{C}\Omega, \mathbb{C}\Omega')$!

Example: points and lines of a square

$G = D_8$ acts on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\Omega' = \{\ell_{12}, \ell_{23}, \ell_{34}, \ell_{14}\}$:



$\Omega \times \Omega'$ splits into orbits R_1 (incident point-line pairs) and R_2 (non-incident point-line pairs).

We define corresponding maps f_1, f_2 from $\mathbb{C}\Omega$ to $\mathbb{C}\Omega'$:

- $f_1(\omega_1) = \ell_{12} + \ell_{14}, f_1(\omega_2) = \ell_{12} + \ell_{23}, \ldots$
- $f_2(\omega_1) = \ell_{23} + \ell_{34}, f_2(\omega_2) = \ell_{14} + \ell_{34}, \ldots$.

Now $\{f_1, f_2\}$ is a basis for $Hom(\mathbb{C}\Omega_1, \mathbb{C}\Omega_2)$.

Characteristic vector of a subset $S \subseteq \Omega$

Characteristic vector χ_S is $(1,1,0,1,0,\ldots)^* \in \mathbb{C}\Omega$, with $(\chi_S)_{\omega_i} = 1$ if $\omega_i \in S$, $(\chi_S)_{\omega_i} = 0$ if $\omega_i \notin S$.

Intersections of two subsets S_1 and S_2

If $S_1, S_2 \subseteq \Omega$, then:

$$\begin{array}{rcl} (\chi_{S_1})^*\chi_{S_2} & = & (1,1,0,1,0,\ldots)(0,1,1,1,0,\ldots)^* \\ & = & 1.0 + 1.1 + 0.1 + 1.1 + 0.0 \\ & = & |S_1 \cap S_2|. \end{array}$$

Decomposition of permutation representation

Suppose G acts transitively on Ω:

$$\mathbb{C}\Omega = \mathbb{C}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp (A_1 \oplus \ldots \oplus A_a) \perp (B_1 \oplus \ldots \oplus B_b) \perp \ldots .$$

Consider any $S \subseteq \Omega$. We can decompose accordingly:

$$\chi_S = \frac{|S|}{|\Omega|}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp (v_{A,1} + \ldots + v_{A,a}) \perp (v_{B,1} + \ldots + v_{B,b}) \perp \ldots$$

Two subsets that are nice with respect to each other

Consider $S, S' \subseteq \Omega$ with:

$$\chi_S = \frac{|S|}{|\Omega|}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp (v_{A,1} + \ldots + v_{A,a}) \perp (\cancel{v_{B,1}} + \ldots + \cancel{v_{B,b}}) + \ldots$$

$$\chi_{S'} = \frac{|S'|}{|\Omega|}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp (\cancel{v'_{A,1}} + \ldots + \cancel{v'_{A,a}}) \perp (v'_{B,1} + \ldots + v'_{B,b}) + \ldots$$

$\forall g \in G$, we have in this case:

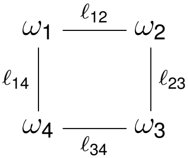
$$\chi_{(S')^g} = \frac{|S'|}{|\Omega|}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp (\cancel{(v'_{A,1})^g} + \ldots + \cancel{(v'_{A,a})^g}) \perp ((v'_{B,1})^g + \ldots + (v'_{B,b})^g) \perp \ldots$$

$$\Longrightarrow |S \cap (S')^g| = (\chi_S)^*(\chi_{(S')^g}) = |S||S'|/|\Omega|.$$

Remember: if G acts transitively on Ω, Ω', then:
$dim(Hom(\mathbb{C}\Omega, \mathbb{C}\Omega')) =$ number of orbits on $\Omega \times \Omega'$!

Example: the square revisited

$G = D_8$ acts on $\Omega_1 = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\Omega_2 = \{\ell_{12}, \ell_{23}, \ell_{34}, \ell_{14}\}$:



$$, \mathbb{C}\Omega_1 = \mathbb{C}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp A \perp B, \mathbb{C}\Omega_2 = \mathbb{C}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \perp A \perp C.$$

- $dim(Hom(\mathbb{C}\Omega_1, \mathbb{C}\Omega_1)) = 3$ ($=$,collinear, opposite points)
- $dim(Hom(\mathbb{C}\Omega_2, \mathbb{C}\Omega_2)) = 3$ ($=$,concurrent and opposite lines)
- $dim(Hom(\mathbb{C}\Omega_1, \mathbb{C}\Omega_2)) = 2$ (incidence and non-incidence)

Our goal

We consider:

- an incidence structure of rank n: with objects of n types (points, lines, planes,...)
- the full automorphism group G of this structure.

Examples: PG(n, q), polar spaces, generalized hexagons,...
We will consider the n permutation representations on points, lines,...

Example: the rank n polar space $W(2n-1,q)$

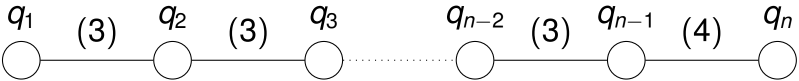
Objects:
$\frac{q^{2n}-1}{q-1}$ points,..., $((q+1)\dots(q^n+1))$ n-spaces or *generators*.
We decompose the permutation representation on generators:

$$\mathbb{C}\Omega = V_0 \perp V_1 \perp \dots \perp V_i \perp \dots \perp V_n,$$

$$\dim(V_i) = q^i \frac{q^{n-i+1}-1}{q^i-1} \frac{q^{n+1-2i}+1}{2} \frac{(q^{2n}-1)\dots(q^{2n+4-2i})}{(q^{2i-2}-1)\dots(q^2-1)}.$$

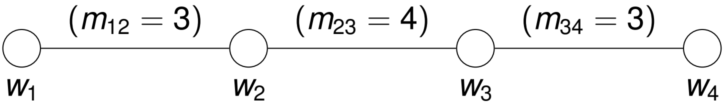
.....But for $q=1$, $dim(V_i)$ just becomes $\binom{n}{i}$!
Many nice incidence structures are buildings, where $q=1$ plays a fundamental role!

Buildings of rank n are certain incidence structures with n types (Tits).
Each finite building has a Coxeter-Dynkin diagram, for instance:

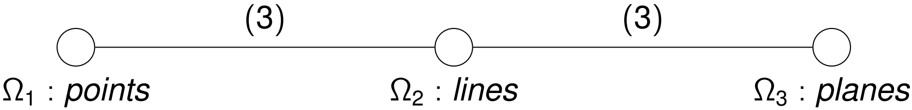


- Each node corresponds with a type (points, lines,planes,...).
- If $x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_n$ of types $1, \ldots, i-1, i+1, \ldots, n$ are incident,
  they are incident with $q_i + 1$ common objects of type i.
- The thin building with $(q_1, \ldots, q_n) = (1, \ldots, 1)$ is constructed from “Coxeter group" itself,
  thick building is more complicated but similar.

Coxeter group W and its thin building



- W is generated by involutions w_i, with $(w_i w_j)^{m_{ij}} = e$ (we draw no line if $m_{ij} = 2 \iff w_i w_j = w_j w_i$),
- maximal parabolic subgroup $P_i := \langle \{w_1, \ldots, w_n\} \setminus \{w_i\} \rangle$,
- objects of type i: cosets wP_i,
- aP_i and bP_j are incident iff $aP_i \cap bP_j \neq \emptyset$,
- decompositions of permutation representation of thick building: in correspondence with that of thin building! (Curtis-Iwahori-Kilmoyer)



We will decompose and write degree for points and lines.

Thin A_3 building= tetrahedron

- $\mathbb{C}\Omega_1 = A_1 \perp B_3$, $\mathbb{C}\Omega_2 = A_1 \perp B_3 \perp C_2$.

Thick A_3 building= projective space PG$(3, q)$

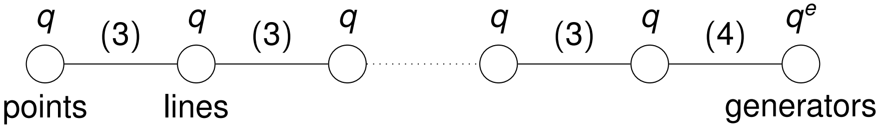
- $\mathbb{C}\Omega_1 = A_1 \perp B_{q^3+q^2+q}$, $\mathbb{C}\Omega_2 = A_1 \perp B_{q^3+q^2+q} \perp C_{q^4+q^2}$.

Nice sets of lines $S \subseteq \Omega_2$ in PG(3, q)

$\mathbb{C}\Omega_2 = A \perp B \perp C$

- Set of lines through one point: $\chi_S \in A \perp B \perp \not{C}$,
- S is k-cover (i.e. every point on k lines of S) $\iff \chi_S \in A \perp \not{B} \perp C$,
- S is Cameron-Liebler line class $\iff \chi_S \in A \perp B \perp \not{C}$.

B_n



- B_n is hyperoctahedral group of size $(2^n n!)$.
- Thick buildings of type B_n are the polar spaces.

Decompositions for (classical) polar spaces

Remember that we just have to look at small group B_n!

$$\begin{aligned}
\mathbb{C}\Omega_1 &= V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \\
\mathbb{C}\Omega_2 &= V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \perp [V^{2,0} \perp V^{2,1} \perp V^{2,2}] \\
&\vdots \\
\mathbb{C}\Omega_{n-1} &= V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \perp [V^{2,0} \perp V^{2,1}] \perp \dots \perp [V^{n-2,0} \perp V^{n-2,1}] \perp [V^{n-1,0} \perp V^{n-1,1}] \\
\mathbb{C}\Omega_n &= V^{0,0} \perp [V^{1,0}] \perp [V^{2,0}] \perp \dots \perp [V^{n-2,0}] \perp [V^{n-1,0}] \perp [V^{n,0}]
\end{aligned}$$

Nice sets of points S in the polar space

- $\chi_S \in V^{0,0} \perp V^{1,0} \perp \cancel{V^{1,1}} \iff S$ is *tight set*,
- $\chi_S \in V^{0,0} \perp \cancel{V^{1,0}} \perp V^{1,1} \iff S$ is k-ovoid
  (i.e. every generator meets it in k points).

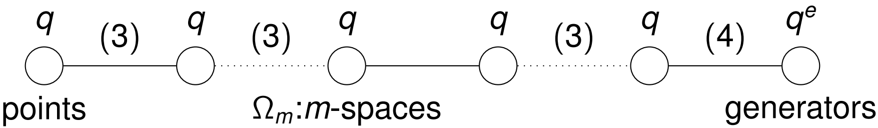
The special subspace $V^{1,0}$

$\mathbb{C}\Omega_m = V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \perp [V^{2,0} \perp V^{2,1} \perp V^{2,2}] \perp \dots$

$V^{1,0}$ corresponds with the *reflection representation* of degree n of Coxeter group B_n.

Obtaining bounds

- Let $p^{1,0} : \mathbb{C}\Omega_m \to V^{1,0}$ denote orthogonal projection.
- For every $S \subseteq \Omega_m : \langle p^{0,0}(\chi_S), p^{0,0}(\chi_S)\rangle \geq 0$,
  with equality iff $\chi_S \in (V^{1,0})^{\perp}$.



$$\mathbb{C}\Omega_m = V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \perp [V^{2,0} \perp V^{2,1} \perp V^{2,2}] \perp \dots$$

- We say $S \subset \Omega_m$ is *partial m-system* if $A \cap B^\perp = \{0\}$ with $A \neq B, \in S$.
- In this case: $\langle p^{1,0}(\chi_S), p^{1,0}(\chi_S)\rangle \geq 0 \iff |S| \leq q^{n-1}.q^e + 1$.
- Note that this *ovoid number* is same bound for every m (Shult-Thas, 1994).
- *m-systems* are the partial m-systems reaching this bound, and hence those with $\chi_S \in (V^{1,0})^\perp$.

Alternative proof for result on *m*-systems (Shult-Thas)

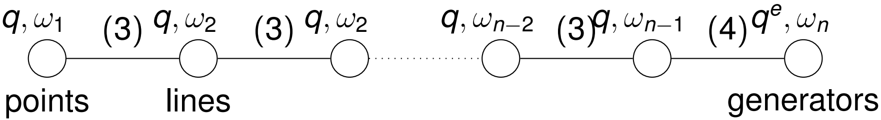
$$\mathbb{C}\Omega_m = V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \perp [V^{2,0} \perp V^{2,1} \perp V^{2,2}] \perp \dots$$

- m-system S: $\chi_S \in V^{0,0} \perp [\cancel{V^{1,0}} \perp V^{1,1}] \perp \dots$.
- Set of m-spaces T of $Q(2n,q), Q^+(2n-1,q), H(2n-1,q^2)$ in $Q^-(2n+1,q), Q(2n,q), H(2n,q^2)$: $\chi_T \in V^{0,0} \perp [V^{1,0} \perp \cancel{V^{1,1}}] \perp [\cancel{V^{2,0}} \perp \cancel{V^{2,1}} \perp \cancel{V^{2,2}}] \perp \dots$.
- $|S \cap T| = |S||T|/|\Omega_m|$.

Alternative proof for 2nd result on *m*-systems (Shult-Thas)!

$\mathbb{C}\Omega_m = V^{0,0} \perp [V^{1,0} \perp V^{1,1}] \perp [V^{2,0} \perp V^{2,1} \perp V^{2,2}] \perp \ldots.$
$\mathbb{C}\Omega_1 = V^{0,0} \perp [V^{1,0} \perp V^{1,1}].$

- *m*-system S: $\chi_S \in V^{0,0} \perp [\cancel{V^{1,0}} \perp V^{1,1}] \perp \ldots.$
- incidence between *m*-spaces and points yields homomorphism $f : \mathbb{C}\Omega_m \to \mathbb{C}\Omega_1$.
- As f is homomorphism: f maps subrepr. into corresponding equivalent subrepr.!
- Point-set $\tilde{S}$ of points in some $\pi \in S$ is given by: $\chi_{\tilde{S}} = f(\chi_S)$.
- $\chi_{\tilde{S}} \in V^{0,0} \perp [\cancel{V^{1,0}} \perp V^{1,1}]$.
- Hence $\tilde{S}$ is a k-ovoid
(i.e. every generator meets $\tilde{S}$ in k points).



Another interesting subspace (exclusively) for generators

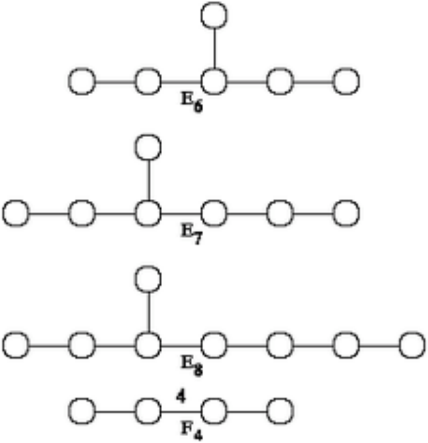
$$\mathbb{C}\Omega_n = V^{0,0} \perp [V^{1,0}] \perp [V^{2,0}] \perp \ldots \perp [V^{n-2,0}] \perp [V^{n-1,0}] \perp [V^{n,0}]$$

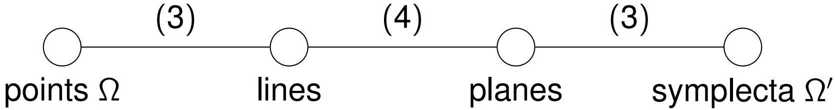
- $V^{n,0}$ corresponds with 1-dim representation ρ of B_n, with:

$$\rho(\omega_1) = \ldots = \rho(\omega_{n-1}) = 1, \rho(\omega_n) = -1.$$

- For partial n-system S, n odd, projection $p^{n,0} \to V^{n,0}$ yields: $\langle p^{n,0}(\chi_S), p^{n,0}(\chi_S) \rangle \geq 0 \iff |S| \leq 1 - \frac{q^{n^2-n}(q^e)^n}{q^{n^2-n}(-1)^n}$.
- $\Longrightarrow$ max. size of partial spread in $H(4n+1, q^2)$ is $q^{2n+1} + 1$.

What about other finite Coxeter groups?



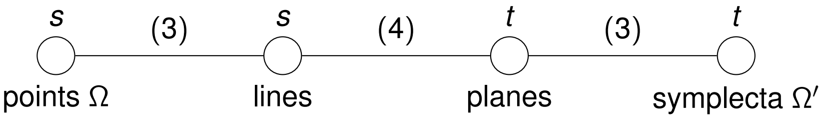


F_4

- $|F_4| = 1152 = 2^7.3^2$.
- Thin: 24 points (Ω), 96 lines, 96 planes, 24 symplecta (Ω′).
- Decompositions of $\mathbb{C}\Omega$ and $\mathbb{C}\Omega'$ in thin case with degrees:

$$\begin{aligned} \mathbb{C}\Omega &= J_1 \perp A_4 \perp B_9 \perp C_8 \perp D_2 \\ \mathbb{C}\Omega' &= J_1 \perp A_4 \perp B_9 \perp C_8' \perp D_2'. \end{aligned}$$

Only the first 3 irreducible representations are equivalent!



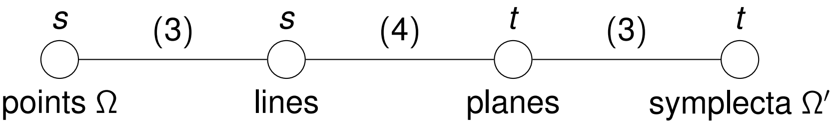
Thick building of type F_4

- $(s,t) = (q,1), (1,q), (q,q), (q,q^2)$ or (q^2,q),
- number of points $|\Omega|$ and of symplecta $|\Omega'|$:

$$\begin{aligned} |\Omega| &= (s^2+s+1)(s^2t+1)(s^2t^2+1)(s^3t^3+1), \\ |\Omega'| &= (t^2+t+1)(t^2s+1)(t^2s^2+1)(t^3s^3+1). \end{aligned}$$

- Decompositions of permutation repr. on points and symplecta:

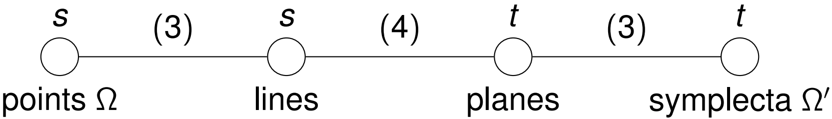
$$\begin{aligned} \mathbb{C}\Omega &= J \perp A \perp B \perp C \perp D \\ \mathbb{C}\Omega' &= J \perp A \perp B \perp C' \perp D'. \end{aligned}$$



$$\begin{aligned}\mathbb{C}\Omega &= J \perp A \perp B \perp C \perp D\\ \mathbb{C}\Omega' &= J \perp A \perp B \perp C' \perp D'.\end{aligned}$$

Nice subsets of points in Ω

- Set of points S in 1 symplecton π:
  consider incidence map $f \in Hom(\mathbb{C}\Omega', \mathbb{C}\Omega)$:
  $\chi_S = f(\chi_\pi) \in \mathrm{Im}(f) = J \perp A \perp B \perp \cancel{C} \perp \cancel{D}$.
- Hence if $\chi_T \in J \perp \cancel{A} \perp \cancel{B} \perp C \perp D$,
  then every symplecton meets T in $|S||T|/|\Omega|$ points.



The special subspace A

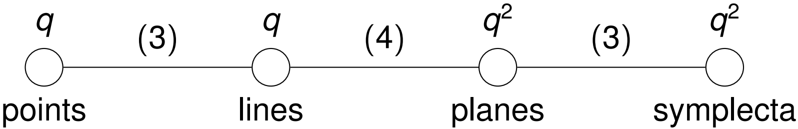
$\mathbb{C}\Omega = J \perp A \perp B \perp C \perp D$.

A corresponds with the *reflection representation* of degree 4 of Coxeter group F_4.

Obtaining bounds

- $S \subset \Omega$ is *partial ovoid* if no 2 points $a, b \in S$ are on common lines with a third point.
- Projection $p : \mathbb{C}\Omega \rightarrow A$ yields for partial ovoids S: $|S| \leq 1 + s^3t^3$, with equality iff $\chi_S \in J \perp \cancel{A} \perp B \perp C \perp D$.

An application for $(s, t) = (q, q^2)$!



Suppose

- T: point set of embedded F_4-building with $(s', t') = (q, q)$: $\chi_T \in J \perp A \perp \cancel{B} \perp \cancel{C} \perp \cancel{D}$.
- S: partial ovoid with $|S| = 1 + s^3t^3 = 1 + q^9$: $\chi_S \in J \perp \cancel{A} \perp B \perp C \perp D$.

$\Longrightarrow |S \cap T| = (\chi_S)^*\chi_T = |S||T|/|\Omega| = q^3 + 1.$

Thank you for your attention!

(Slides will appear on `http://cage.ugent.be/~fvanhove`)