Complex remainder Tauberian theorems

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Wiener-Ikehara theorem

Theorem (Wiener-Ikehara)

Let S be a non-decreasing function with support on $[0, \infty)$ and let

$$\mathcal{L}\{\mathrm{d}S; s\} := \int_0^\infty e^{-sx}\mathrm{d}S(x) \text{ be convergent for } \Re e\, s > 1.$$

Suppose that there exists a such that

$$G(s) := \mathcal{L}\{\mathrm{d}S; s\} - \frac{a}{s-1}$$

admits an analytic extension beyond the line $\Re e\, s = 1$. Then

$$S(x) = ae^x + o(e^x).$$

Weakening of hypothesis analytic extension

The hypothesis analytic extension is not necessary. Each of the following conditions is sufficient:

1. Continuous extension,
2. L^1_{loc}-extension,
3. Extension to a local pseudo-function

It turns out that the latter hypothesis is also necessary.

Local pseudo-functions

Definition

A distribution f is locally a pseudo-function if

$$\lim_{h\to\infty} \left\langle f(t), e^{iht}\phi(t) \right\rangle = 0, \quad \forall \phi \in \mathcal{D}.$$

Definition

A function G which is analytic on the half-plane $\Re e\, s > 1$ admits local pseudo-function behavior on the line $\Re e\, s = 1$ if for each $\phi \in \mathcal{D}$,

$$\lim_{\sigma\to 1^+} \int G(\sigma + it)\phi(t)\mathrm{d}t = \langle g(t), \phi(t) \rangle ,$$

where g is locally a pseudo-function.

Remainders

Question: What are the (minimal) requirements to obtain remainders in the Wiener-Ikehara theorem? We work under the following hypotheses:

Theorem (Model theorem)

Let S be a non-decreasing function and T a differentiable function satisfying $T'(x) \leq Ce^{x}$ with both functions supported on $[0, \infty)$. Suppose that

$$G(s) := \int_{0^-}^{\infty} e^{-su} \mathrm{d}S(u) - \mathrm{d}T(u)$$

has "good behavior" (to be determined). Then

$$S(x) = T(x) + \mathit{remainder}.$$

Remark regarding T

Remark

One does not need to fix the bound $T'(x) \leq Ce^x$. Better bounds on T' will generally lead to better remainders.

Even for the $o(e^x)$-remainder in the classical Wiener-Ikehara theorem, one can apparently weaken the hypotheses on G if T behaves well enough.

The remainder $O(e^x)$

Definition

A distribution f is locally a pseudo-measure on (a, b) if

$$\left\langle f(t), e^{iht}\phi(t)\right\rangle = O(1), \quad \forall \phi \in \mathcal{D}(a, b).$$

Theorem

Let S be a non-decreasing function having support on $[0, \infty)$. Suppose that

$$G(s) := \int_{0^-}^{\infty} e^{-su} \mathrm{d}S(u)$$

has local pseudo-measure behavior at $s = 1$. Then

$$S(x) = T(x) + O(e^x)$$

The remainder $O(e^x/x^{-\beta})$

Theorem

Let S be a non-decreasing function and T a differentiable function satisfying $T'(x) \leq Ce^x$ with both functions supported on $[0,\infty)$. Suppose that

$$G(s) := \int_{0^-}^{\infty} e^{-su} \mathrm{d}S(u) - \mathrm{d}T(u)$$

admits a C^N-extension to the line $\Re e\, s = 1$ and

$$G^{(N)}(1+it) = O(|t|^{\gamma}) \quad (\gamma > 1).$$

Then

$$S(x) = T(x) + O(e^x x^{-N/(\gamma+1)}).$$

By using Hölder continuity one can generalize this theorem to allow N to be a non-integer .

Special case: the remainder $O(e^x x^{-n})$ for all n

Theorem

Let S be a non-decreasing function and T a differentiable function satisfying $T'(x) \leq Ce^x$ with both functions supported on $[0, \infty)$. Suppose that

$$G(s) := \int_{0^-}^{\infty} e^{-su} \mathrm{d}S(u) - \mathrm{d}T(u)$$

admits a C^∞-extension to the line $\Re e\, s = 1$ and $G^{(n)}(1 + it) = O_n(|t|^\gamma)$ for some γ and all n. Then

$$S(x) = T(x) + O(e^x x^{-n}) \quad \text{for all } n.$$

This theorem is if-and-only-if.

Subexponential remainders

Theorem

Let S be a non-decreasing function and T a differentiable function satisfying $T'(x) \leq Ce^{x}$ with both functions supported on $[0, \infty)$. Suppose that

$$G(s) := \int_{0^-}^{\infty} e^{-su} \mathrm{d}S(u) - \mathrm{d}T(u)$$

admits a C^{∞}-extension to the line $\Re e\, s = 1$ such that

$$\left| G^{(n)}(1 + it) \right| \leq CA^{n} n! (\log |t| + 2)^{n}, \quad \text{for all } n.$$

Then there exists $c > 0$ such that

$$S(x) = T(x) + O(e^{x - c\sqrt{x}}).$$

Associated functions

Definition

Let M_n be a positive, increasing sequence. Its associated function is

$$M(t) = \sup_p \log\left(\frac{t^p}{M_p}\right), \quad t > 0.$$

Example: if $M_n = n^{n\beta}$, then $M(t) = t^{1/\beta}$.

Subexponential remainders

Theorem

Let S be a non-decreasing function and T a differentiable function satisfying $T'(x) \leq Ce^x$ with both functions supported on $[0, \infty)$. Let M_n and N_n be positive, non-decreasing sequences. Let G be as before and suppose that G admits a C^∞-extension to the line $\Re e\, s = 1$ such that

$$\left| G^{(n)}(1 + it) \right| \leq CA^n M_n (N^{-1}(\log|t| + 2))^n, \quad \text{for all } n,$$

where N^{-1} is the inverse of the associated function to the sequence N_n. If N' has at most polynomial growth then there exists $c > 0$ such that

$$S(x) = T(x) + O(e^{x - cV(cx)}),$$

where V is the associated function of the sequence $M_n N_n$.