Complex Tauberian theorems for Laplace transforms

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Wiener-Ikehara theorem

Theorem (Wiener-Ikehara)

Let S be a non-decreasing function with support on $[0,\infty)$ and let

$$\mathcal{L}\{\mathrm{d}S;s\} := \int_0^\infty e^{-sx}\mathrm{d}S(x) \text{ be convergent for } \Re e\, s > 1.$$

Suppose that there exists a such that

$$G(s) := \mathcal{L}\{\mathrm{d}S;s\} - \frac{a}{s-1}$$

admits an analytic extension beyond the line $\Re e\, s = 1$. Then

$$S(x) = ae^x + o(e^x).$$

Weakening of hypothesis analytic extension

The hypothesis analytic extension is not necessary. Each of the following conditions is sufficient:

1. Continuous extension,
2. L^1_{loc}-extension,
3. Extension to a local pseudo-function

It turns out that the latter hypothesis is also necessary.

Local pseudo-functions

Definition

A tempered distribution $f \in \mathcal{S}'$ is a pseudo-function if $\hat{f} \in C_0$. A distribution f is locally a pseudo-function if it coincides on each finite interval with a pseudo-function.

Characterization of local pseudo-functions

Proposition

A distribution f is locally a pseudo-function iff

$$\lim_{h\to\infty} \left\langle f(t), e^{iht}\phi(t) \right\rangle = 0, \quad \forall \phi \in \mathcal{D}.$$

Definition

A distribution f is locally a pseudo-measure if

$$\left\langle f(t), e^{iht}\phi(t) \right\rangle = O(1), \quad \forall \phi \in \mathcal{D}.$$

Local pseudo-function behavior

Definition

A function G which is analytic on the half-plane $\Re e\, s > 1$ admits local pseudo-function behavior on the line $\Re e\, s = 1$ if for each $\phi \in \mathcal{D}$,

$$\lim_{\sigma \to 1^+} \int G(\sigma + it)\phi(t)\mathrm{d}t = \langle g(t), \phi(t) \rangle ,$$

where g is locally a pseudo-function.

Slowly decreasing functions

Definition

A function S is slowly decreasing if for each $\varepsilon > 0$, there exists $\delta > 0$ and N such that

$$S(x+h) - S(x) \geq -\varepsilon, \quad \forall x > N \text{ and } 0 < h < \delta.$$

Tauberian theorem

Theorem

Let $\tau \in L^1_{loc}(\mathbb{R})$ *be such that* $\operatorname{supp}\tau \subseteq [0,\infty)$ *and slowly decreasing. Suppose that*

$$G(s) := \mathcal{L}\{\tau; s\} \text{ converges for } \Re e\, s > 0$$

and admits local pseudo-measure behavior near $s = 0$*, then*

$$\tau(x) = O(1), \quad x \to \infty.$$

Sketch of proof (for two-sided condition)

1. The condition local pseudo-measure behavior translates to

$$\left\langle \tau(x+h), \hat{\phi}(x) \right\rangle = O(1), \quad \forall \phi \in \mathcal{D}(-\lambda, \lambda).$$

2. Use Tauberian condition to find $\tau(x) = O(1)$.

Theorem with local pseudo-function behavior

Theorem

Let $\tau \in L^1_{loc}(\mathbb{R})$ *be such that* $\operatorname{supp} \tau \subseteq [0, \infty)$ *and slowly decreasing. Suppose that*

$$G(s) := \mathcal{L}\{\tau; s\} \text{ converges for } \Re e\, s > 0$$

and admits local pseudo-function behavior on the line $\Re e\, s = 0$*, then*

$$\tau(x) = o(1), \quad x \to \infty.$$

Sketch of proof

1. The Banach-Steinhaus theorem ensures that local pseudo-function behavior translates to

$$\langle \tau(x+h), \phi(x) \rangle = O(1), \quad \forall \phi \in \mathcal{S}.$$

2. Choose a positive $\phi \in \mathcal{D}(0, \delta)$ with $\int \phi = 1$,

$$\tau(h) \leq \int \tau(x+h)\phi(x)\mathrm{d}x + \varepsilon \leq 2\varepsilon$$

3. Lower bound is similar.

Very slowly decreasing functions

Definition

A function S is very slowly decreasing if for each $\varepsilon > 0$, there exists N such that

$$S(x+h) - S(x) \geq -\varepsilon, \quad \forall x > N \text{ and } 0 < h < 1.$$

Version of theorem for very slowly decreasing functions

Theorem

Let $\tau \in L^1_{loc}(\mathbb{R})$ *be such that* $\operatorname{supp}\tau \subseteq [0,\infty)$ *and slowly decreasing. Suppose that*

$$G(s) := \mathcal{L}\{\tau; s\} \text{ converges for } \Re e\, s > 0$$

and admits local pseudo-function behavior near $s = 0$*, then*

$$\tau(x) = o(1), \quad x \to \infty.$$

Finite form versions

Theorem

Let $\rho \in L^1_{loc}(\mathbb{R})$ *be such that* $\limsup_{x\to\infty} |\rho(x)| := M$ *and vanishes on* $(-\infty, 0)$. *Suppose that there is* $\lambda > 0$ *such that*

$$\frac{\mathcal{L}\{\rho; s\}}{s}$$

has local pseudo-function boundary behavior on $i(-\lambda, \lambda)$. *Then*

$$\limsup_{x\to\infty} \left| \int_0^x \rho(u)\mathrm{d}u \right| \leq \frac{M\pi}{2\lambda}.$$

Moreover the constant $\pi/2$ *cannot be improved.*

References

G. Debruyne, J. Vindas, *Complex Tauberian theorems for Laplace transforms with local pseudofunction boundary behavior*, arxiv 1604.05069