Optimality of Tauberian theorems

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Ingham-Karamata theorem

Theorem (Ingham, Karamata, 1934)

Let $\rho \in L^\infty$ *be supported on the positive half-axis. Suppose that*

$$\frac{\mathcal{L}\{\rho; s\}}{s} = \frac{1}{s}\int_0^\infty e^{-su}\rho(u)\mathrm{d}u$$

admits an analytic continuation beyond $\operatorname{Re} s = 0$*, then*

$$\int_0^x \rho(u)\mathrm{d}u = o(1), \quad x \to \infty.$$

Applications

1. Number theory: Newman used a simpler version of the Ingham-Karamata theorem for his proof of the PNT (1980).
2. Operator theory
3. Probability theory
4. Semigroup theory
5. . . .

An example in semigroup theory

Theorem (Arendt, Batty, 1988)

Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X with generator A. Suppose that the spectrum satisfies $\sigma(A) \cap i\mathbb{R} = \emptyset$, then

$$\|T(t)A^{-1}\| \to 0, \quad t \to \infty.$$

Abstract Cauchy problem

The Arendt-Batty theorem implies that the classical solutions of the abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geq 0, \\ x(0) = x_0, & x_0 \in X, \end{cases}$$

given by $x(t) = T(t)x_0$, $t \geq 0$, $x_0 \in D(A)$, converge to 0 as $t \to \infty$ if A satisfies the hypotheses of the Arendt-Batty theorem.

A quantified version for semigroups

Theorem (Batty, Duyckaerts, 2008)

Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on the Banach space X whose generator A satisfies $\sigma(A) \cap i\mathbb{R} = \emptyset$. Suppose that $M\colon \mathbb{R}_+ \to (0,\infty)$ is a non-decreasing continuous function such that the resolvent satisfies $\|R(is,A)\| = O(M(|s|))$ as $|s| \to \infty$. Then

$$\|T(t)A^{-1}\| = O\left(\frac{1}{M_{\log}^{-1}(ct)}\right), \quad t \to \infty,$$

for some $c > 0$, where
$M_{\log}(s) = M(s)(\log(1+s) + \log(1+M(s)))$.

Quantified Ingham-Karamata theorem

Theorem (Stahn, 2017)

Let M, K be continuous non-decreasing functions. Let $\rho \in L^\infty$ such that $\mathcal{L}\{\rho; s\}$ admits an analytic extension to

$$\Omega_M = \left\{ s \in \mathbb{C} : \operatorname{Re} s > -\frac{1}{M(|\operatorname{Im} s|)} \right\},$$

where it satisfies the bound

$$\sup_{s \in \Omega_M} \frac{|s\mathcal{L}\{\rho; s\}|}{K(|\operatorname{Im} s|)} < \infty,$$

then for all $c < 1$

$$\int_0^x \rho(u)\mathrm{d}u = O\left(\frac{1}{M_K^{-1}(cx)}\right), \quad x \to \infty,$$

where $M_K(s) = M(s)(\log(1 + s) + \log(1 + K(s)))$.

Optimality of the Tauberian theorem

Theorem (Debruyne, Seifert, 2018)

Let M, K, r be non-decreasing functions and assume that M and K are continuous. Suppose $M_K(s) = O(\exp(\alpha s))$ for some $\alpha > 0$. If

$$|f(x)| = O\big(r(x)^{-1}\big), \quad x \to \infty,$$

for every Lipschitz continuous function whose Laplace transform extends analytically to the region Ω_M and is $O(|K(\operatorname{Im} s)|)$ there, then

$$r(x) = O\big(M_K^{-1}(x)\big), \quad x \to \infty.$$

A taste of the proof: sketch of the main idea

Let X be the Banach space consisting of all (bounded) Lipschitz continuous functions whose Fourier transform extends analytically to Ω_M where it is $O(|K(\operatorname{Im} s)|)$, topologized canonically via the seminorm

$$\|g\|_X = \|g\|_{L^\infty} + \|g'\|_{L^\infty} + \sup_{s \in \Omega_M} \frac{|s\widehat{g}(s)|}{1 + K(|\operatorname{Im} s|)},$$

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and let Y consist of all the functions $g \in X$ such that additionally $g(x) = O(r(x)^{-1})$, topologized via

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By hypothesis, $X = Y$ and Y is continuously embedded in X. By the open mapping theorem, there is C such that

$$\sup |g(x)|r(x) \leq C\|g\|_X, \quad g \in X.$$

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Now choose $g(x) = g_{y,\lambda}(x) := f(\lambda(y - x))$ for some appropiate f, then roughly speaking, one obtains

$$V(y) \ll \lambda + \frac{e^{y/M(\lambda)}}{K(\lambda)}.$$

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Now choose λ optimal (dependent of y).

Optimality for semigroups

Theorem (Debruyne, Seifert, 2018)

Let M be a non-decreasing continuous function satisfying $M(s + c/M(s)) \geq cM(s)$ and $M(s) = O(\exp(\alpha s))$ for some $\alpha, c > 0$. There exists a complex Banach space X and a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on X whose generator A satisfies $\sigma(A) \cap i\mathbb{R} = \emptyset$, $\|R(is, A)\| = O(M(|s|))$ as $|s| \to \infty$ and

$$\limsup_{t\to\infty} \left\| M_{\log}^{-1}(t) T(t) A^{-1} \right\| > 0.$$

References

- G. Debruyne, D. Seifert, *Optimal decay of functions and one-parameter operator semigroups.* Submitted, available online at arXiv: 1804.02374.
- G. Debruyne, D. Seifert, *Optimality of the quantified Ingham-Karamata theorem for operator semigroups with general resolvent growth.* To be finished soon