# Optimality in Tauberian theorems

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# Tauberian Theory

**Tauberian theory:** Extracting asymptotic information from integral transforms

$$\tau(t) \underset{\textbf{Tauberian theory}}{\overset{\textbf{Integral transforms}}{\rightleftarrows}} \int_0^\infty e^{-st}\tau(t)dt, \quad \int_{-\infty}^{\infty} \frac{\tau(t)}{t+z}dt, \quad \ldots$$

# The Ingham-Karamata theorem

**Theorem (Ingham, Karamata, 1934)**

*Let $\tau : \mathbb{R}_+ \to \mathbb{R}$ be such that $\tau(x) + Ax$ is non-decreasing for certain $A > 0$. Suppose that*

$$\mathcal{L}\{\tau; s\} = \int_0^\infty e^{-su}\tau(u)\mathrm{d}u$$

*converges for* $\operatorname{Re} s > 0$ *and admits an analytic continuation beyond* $\operatorname{Re} s = 0$, *then*

$$\tau(x) = o(1), \quad x \to \infty.$$

# An application: short proof of the PNT

**Ingredients:**

- $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ admit an meromorphic extension beyond $\operatorname{Re} s = 1$ with a unique simple pole at $s = 1$ with residue 1.
- $\zeta(1 + it) \neq 0$

Let

$$\psi_1(x) := \sum_{n \leq x} \frac{\Lambda(n)}{n}$$

We aim to show that

$$\psi_1(x) = \log x - \gamma + o(1),$$

where $\gamma$ is the Euler-Mascheroni constant.

# Proof PNT (continued)

We set

$$\tau(x) := \sum_{n \leq e^x} \frac{\Lambda(n)}{n} - x + \gamma.$$

Its Laplace transform is

$$-\frac{\zeta'(s+1)}{s\zeta(s+1)} - \frac{1}{s^2} + \frac{\gamma}{s}.$$

From the ingredients, it follows that $\tau$ satisfies all the hypotheses for Ingham-Karamata, thus

$$\tau(x) = o(1).$$

# Research avenues for Ingham-Karamata

- Weakening the boundary behavior on $\mathcal{L}\{\tau; s\}$.
- Other and more general Tauberian conditions/Allow more general singularities on integral transform.
- Obtaining stronger decay estimates on $\tau$ (with stronger information on the Laplace transform).

# Model theorem: a quantified version

**Theorem**

*Let $N \in \mathbb{N}$, $M > -1$ and $\tau : \mathbb{R}_+ \to \mathbb{R}$ be such that $\tau(x) + Ax$ is non-decreasing for certain $A > 0$. Suppose that*

$$\mathcal{L}\{\tau; s\} = \int_0^\infty e^{-su}\tau(u)\mathrm{d}u$$

*converges for $\operatorname{Re} s > 0$ and admits an $N$ times differentiable extension $g(t) := \mathcal{L}\{\tau; it\}$ to $\operatorname{Re} s = 0$, satisfying*

$$\left|g^{(N)}(t)\right| \ll (1 + |t|)^M, \quad t \in \mathbb{R},$$

*then*

$$\tau(x) \ll x^{-N/(M+2)}, \quad x \to \infty.$$

**Question:** Is the decay rate optimal?

# Structure of the optimality theorem

**Theorem (D., 2018)**

*Suppose that all functions* $\tau$ *satisfying the hypotheses of the model theorem admit the decay rate*

$$\tau(x) \ll \frac{1}{V(x)}, \quad x \to \infty,$$

*then*

$$V(x) \ll x^{N/(M+2)}, \quad x \to \infty.$$

# The key proof idea

Collect the functions $\tau$ who satisfy the (more restrictive) hypotheses of the model theorem into a Banach space $X_1$, topologized via

$$\|\tau\|_1 = \sup_{x\geq 0} |\tau'(x)| + \sup_{t\in\mathbb{R}} \frac{|g^{(N)}(t)|}{(1+|t|)^M}.$$

Collect the functions $\tau$ which additionally satisfy the decay rate $1/V(x)$ in another Banach space $X_2$, topologized via

$$\|\tau\|_2 = \|\tau\|_1 + \sup_{x\geq 0} |\tau(x)V(x)|.$$

# The key proof idea: open mapping theorem

Consider the canonical inclusion mapping $\iota : X_2 \to X_1$, which is clearly continuous.
If $V(x)$ is an acceptable decay rate for the model theorem, then $\iota$ is *surjective* and therefore by the **open mapping theorem** an open mapping, that is, $\iota^{-1}$ is also continuous:

$$\sup_{x\geq 0} |\tau(x)V(x)| \ll \sup_{x\geq 0} |\tau'(x)| + \sup_{t\in\mathbb{R}} \frac{|g^{(N)}(t)|}{(1+|t|)^M}.$$

The rest of the proof consists in considering the families $\tau_{y,\lambda}(x) := \kappa(\lambda(x-y))$ for a well-chosen function $\kappa$ and optimizing the parameters $y$ and $\lambda$.

# The unquantified Ingham-Karamata theorem

**Theorem (D.-Vindas, 2018)**

*Let* $-1 < \alpha < 0$*. Suppose every function who satisfies the hypotheses of the unquantified Ingham-Karamata theorem with even an analytic extension to the half-plane* $\operatorname{Re} s > -\alpha$ *satisfies* $\tau(x) \ll V(x)$*, then*

$$V(x) \ll 1.$$

For a constructive proof (Broucke-D.-Vindas, 2021)

# A general quantified theorem optimality result

**Theorem (D.-Seifert, 2019)**

*Let* $M, K : \mathbb{R}_+ \to \mathbb{R}$ *be non-decreasing positive functions. Let*

$$M_K(x) := M(x)(\log(1+x) + \log(1+K(x)) \ll \exp(\alpha x)$$

*for some* $\alpha > 0$. *Suppose that for all functions* $\tau$ *for which* $\tau(x) + Ax$ *is non-decreasing and whose Laplace transforms admit analytic extensions to*

$$\Omega_M = \left\{ \sigma + it : \sigma > -\frac{1}{M(|t|)} \right\}.$$

*where they satisfy the bound* $K(|t|)/(1+|t|)$, *satisfy the decay estimate*

$$\tau(x) \ll 1/V(x).$$

*Then*

$$V(x) \ll M_K^{-1}(x).$$