Recent results on the Fatou-Riesz theorem

Gregory Debruyne

Universiteit Gent

August 14, 2017

Original Fatou-Riesz theorem

Theorem (Fatou, 1906)

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

satisfy the Tauberian condition $a_n = o(1)$ *(such that in particular* f *is analytic in the unit disc). Suppose that* f *admits an analytic extension at* $z = 1$*, then*

$$\sum_{n=0}^{\infty} a_n = f(1).$$

A Fatou-Riesz type Tauberian theorem

Theorem (Ingham, Karamata, 1934)

Let $\rho \in L^{\infty}$ be supported on the positive half-axis. Suppose that

$$\mathcal{L}\{\rho; s\} = \int_0^{\infty} e^{-su}\rho(u)\mathrm{d}u \quad \text{converges for } \Re e\, s > 0,$$

and that $\mathcal{L}\{\rho; s\}/s$ admits an analytic continuation beyond $\Re e\, s = 0$, then

$$\int_0^{x} \rho(u)\mathrm{d}u = o(1), \quad x \to \infty.$$

Slowly decreasing functions

Definition

A function S is slowly decreasing if for each $\varepsilon > 0$, there exists $\delta > 0$ and N such that

$$S(x+h) - S(x) \geq -\varepsilon, \quad \forall x > N \text{ and } 0 < h < \delta.$$

Definition

A function S is very slowly decreasing if for each $\varepsilon > 0$, there exists N such that

$$S(x+h) - S(x) \geq -\varepsilon, \quad \forall x > N \text{ and } 0 < h < 1.$$

Other version Ingham-Karamata theorem

Theorem

Let $\tau \in L^1_{loc}(\mathbb{R})$ *be slowly decreasing such that* $\operatorname{supp} \tau \subseteq [0, \infty)$*. Suppose that*

$$G(s) := \mathcal{L}\{\tau; s\} \text{ converges for } \Re e\, s > 0$$

and admits an analytic extension beyond $\Re e\, s = 0$*, then*

$$\tau(x) = o(1), \quad x \to \infty.$$

Other version Fatou-Riesz theorem

Theorem

Let $\tau \in L^1_{loc}(\mathbb{R})$ *be very slowly decreasing such that* $\operatorname{supp} \tau \subseteq [0, \infty)$. *Suppose that*

$$G(s) := \mathcal{L}\{\tau; s\} \text{ converges for } \Re e\, s > 0$$

and admits an analytic extension at $s = 0$, *then*

$$\tau(x) = o(1), \quad x \to \infty.$$

Some areas of applications

These type of theorems have numerous applications in

1. Number theory
2. Operator theory
3. Semigroup theory

Weakening of hypothesis analytic extension

The hypothesis analytic extension is not necessary. Each of the following conditions is sufficient:

1. Continuous extension,
2. L^1_{loc}-extension,
3. Extension to a local pseudo-function

It turns out that the latter hypothesis is also necessary.

Local pseudo-functions

Definition

A tempered distribution $f \in \mathcal{S}'$ is a pseudo-function if $\hat{f} \in C_0$. A distribution f is locally a pseudo-function if it coincides on each finite interval with a pseudo-function.

Characterization of local pseudo-functions

Proposition

A distribution f is locally a pseudo-function iff

$$\lim_{h\to\infty}\left\langle f(t), e^{iht}\phi(t)\right\rangle = 0, \quad \forall\phi\in\mathcal{D}.$$

Definition

A distribution f is locally a pseudo-measure if

$$\left\langle f(t), e^{iht}\phi(t)\right\rangle = O(1), \quad \forall\phi\in\mathcal{D}.$$

Local pseudo-function boundary behavior

Definition

A function G which is analytic on the half-plane $\Re e\, s > 1$ admits local pseudo-function behavior on the line $\Re e\, s = 1$ if for each $\phi \in \mathcal{D}$,

$$\lim_{\sigma \to 1^+} \int G(\sigma + it)\phi(t)\mathrm{d}t = \langle g(t), \phi(t) \rangle ,$$

where g is locally a pseudo-function.

Boundedness result

Theorem (Debruyne, Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ *be such that* $\operatorname{supp} \tau \subseteq [0, \infty)$ *and slowly decreasing. Suppose that*

$$G(s) := \mathcal{L}\{\tau; s\} \text{ converges for } \Re e\, s > 0$$

and admits local pseudo-measure behavior at $s = 0$*, then*

$$\tau(x) = O(1), \quad x \to \infty.$$

Theorem with local pseudo-function behavior

Theorem (Debruyne, Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ *be such that* $\operatorname{supp} \tau \subseteq [0, \infty)$ *and slowly decreasing. Suppose that*

$$G(s) := \mathcal{L}\{\tau; s\} \textit{ converges for } \Re e\, s > 0$$

and admits local pseudo-function behavior on the line $\Re e\, s = 0$*, then*

$$\tau(x) = o(1), \quad x \to \infty.$$

Version for very slowly decreasing functions

Theorem (Debruyne,Vindas, 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be very slowly decreasing such that $\operatorname{supp} \tau \subseteq [0, \infty)$. Suppose that

$$G(s) := \mathcal{L}\{\tau; s\} \text{ converges for } \Re e\, s > 0$$

and admits local pseudo-function behavior at $s = 0$, then

$$\tau(x) = o(1), \quad x \to \infty.$$

Finite form versions: two-sided condition

Theorem (Debruyne, Vindas, 2017)

Let $\rho \in L^1_{loc}(\mathbb{R})$ be such that $\limsup_{x\to\infty} |\rho(x)| := M$ and vanishes on $(-\infty, 0)$. Suppose that there is $\lambda > 0$ such that

$$\frac{\mathcal{L}\{\rho; s\}}{s}$$

has local pseudo-function boundary behavior on $i(-\lambda, \lambda)$. Then

$$\limsup_{x\to\infty} \left| \int_0^x \rho(u)\mathrm{d}u \right| \leq \frac{M\pi}{2\lambda}.$$

Moreover the constant $\pi/2$ cannot be improved.

Finite form versions: one-sided condition

Theorem (Debruyne, Vindas, 2017)

Let $\rho \in L^1_{loc}(\mathbb{R})$ be such that $\liminf_{x\to\infty} \rho(x) := -M$ and vanishes on $(-\infty, 0)$. Suppose that there is $\lambda > 0$ such that

$$\frac{\mathcal{L}\{\rho; s\}}{s}$$

has local pseudo-function boundary behavior on $i(-\lambda, \lambda)$. Then

$$\limsup_{x\to\infty} \left| \int_0^x \rho(u)\mathrm{d}u \right| \leq \frac{M\pi}{\lambda}.$$

Moreover the constant π cannot be improved.

References

- G. Debruyne, J. Vindas, *Complex Tauberian theorems for Laplace transforms with local pseudofunction boundary behavior,* J. Anal. Math., to appear (preprint: arXiv:1604.05069)
- G. Debruyne, J. Vindas, *Optimal Tauberian constant in the Fatou-Riesz Tauberian theorem for Laplace transforms,* preprint: arXiv:1705.00667.