The saddle-point method for general partition functions

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Joint work with Gérald Tenenbaum

Partition problem

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Given a set of summands $\Lambda \subset \mathbb{N}$, in how many ways can a natural number n be represented as the sum of summands in Λ?

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We do not care about the order of the summands. For instance, the sums $3+2$ and $2+3$ represent the same partition.
We denote the answer as $p_\Lambda(n)$.

The classical partitions

Theorem (Hardy,Ramanujan, 1918)

$$p_{\mathbb{N}}(n) \sim \frac{\mathrm{e}^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

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In 1937, Rademacher obtained an exact formula, that is, a convergent series expansion.

Admissible sets of summands

Definition

Let $A \in \mathbb{R}$ and

$$L_{\Lambda}(z) := \sum_{m \in \Lambda} m^{-z},$$

we define the *class* $\mathcal{C}(A)$ comprising those subsets Λ fulfilling the following conditions:

(a) $\gcd(\Lambda) = 1$;

(b) L_{Λ} may be meromorphically continued to the closed half-plane $\operatorname{Re} z \geqslant -\varepsilon$ for suitable $\varepsilon > 0$;

(c) this continuation presents a unique simple pole at $z = \sigma_c(\Lambda) > 0$ with residue A;

(d) we have $|L_{\Lambda}(-\varepsilon + it)| \ll \mathrm{e}^{a|t|}$ $(t \in \mathbb{R})$ for some $a < \pi/2$.

An approximation for $p_\Lambda(n)$

Theorem (D., Tenenbaum, 2020)

Let $A \in \mathbb{R}$ *and* $\Lambda \in \mathcal{C}(A)$. *Then*

$$p_\Lambda(n) \sim \mathfrak{b} \mathrm{e}^{\mathfrak{c} n^{\alpha/(\alpha+1)}} / n^{\mathfrak{h}} \qquad (n \to \infty),$$

where $\alpha := \sigma_c(\Lambda)$, $\mathfrak{h} := (1 - L_\Lambda(0) + \alpha/2)/(\alpha + 1)$ *and*

$$\mathfrak{a} := \{A\Gamma(1+\alpha)\zeta(1+\alpha)\}^{1/(\alpha+1)}, \quad \mathfrak{b} := \frac{\mathrm{e}^{L'_\Lambda(0)} \mathfrak{a}^{-L_\Lambda(0)+1/2}}{\sqrt{2\pi(1+\alpha)}},$$

$$\mathfrak{c} := \mathfrak{a}(1 + 1/\alpha).$$

Admissible sets of summands

Definition

We define the *subclass* $\mathcal{D}(A)$ of $\mathcal{C}(A)$ comprising those subsets Λ satisfying the extra conditions:

(e) L_Λ may be meromorphically continued to $\mathbb{C}$;

(f) for suitable $R_N \to \infty$ and some $a < \pi/2$, we have

$$|L_\Lambda(-R_N + it)| \ll \exp(a|t|) \quad (t \in \mathbb{R},\ N \to \infty)$$

(g) for all $q \geqslant 2$ the set $\Lambda \setminus q\mathbb{N}$ is infinite.

An asymptotic formula

Theorem (D, Tenenbaum, 2020)

Let $A \in \mathbb{R}$ *and* $\Lambda \in \mathcal{D}(A)$*, then there exist constants* $\gamma_{j,h}$ $((j,h) \in \mathbb{N}^2)$ *such that for each* $N \geqslant 1$,

$$p_\Lambda(n) = \frac{\mathfrak{b} e^{\mathfrak{c} n^{\alpha/(\alpha+1)}}}{n^{\mathfrak{h}}} \left\{ 1 + \sum_{\substack{j+h \geqslant 1 \\ j\alpha + h \leqslant N(\alpha+1)}} \frac{\gamma_{j,h}}{n^{(j\alpha+h)/(\alpha+1)}} + O\Big(\frac{1}{n^N}\Big) \right\},$$

with $\alpha, \mathfrak{b}, \mathfrak{c}$ *and* $\mathfrak{h}$ *as before.*

A corollary

Corollary (D., Tenenbaum, 2020)

For each $f \in \mathbb{Z}[x]$, *for which* $\Lambda := f(\mathbb{N} \cup \{0\}) \subseteq \mathbb{N}$, f *is injective on* $\mathbb{N} \cup \{0\}$ *and such that* f *does not vanish identically* $\bmod p$ *for any prime* p, *we have*

$$p_\Lambda(n) \sim \frac{\mathfrak{b}_f e^{\mathfrak{c}_f n^{1/(k+1)}}}{n^{(k+1+2a_1/a_0)/(2k+2)}} \left\{ 1 + \sum_{h \geqslant 1} \frac{c_{f,h}}{n^{h/(k+1)}} \right\},$$

where

$$\mathfrak{a}_f = \left\{ k^{-1} a_0^{-1/k} \Gamma(1 + k^{-1}) \zeta(1 + k^{-1}) \right\}^{k/(k+1)},$$

$$\mathfrak{b}_f = \frac{\mathfrak{a}_f^{a_1/a_0 k} a_0^{-1/2 + a_1/a_0 k} \prod_{j=1}^k \Gamma(-\alpha_j)}{(2\pi)^{(k+1)/2} \sqrt{1 + 1/k}}, \quad \mathfrak{c}_f = (k+1)\mathfrak{a}_f,$$

for a polynomial f *of degree* k, *where* a_0 *is the coefficient of the dominant term*, a_1 *that of* n^{k-1} *and the* α_j *are the zeros of* f.

Generating functions

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Saddle-point method: We search for solutions of

$$n + F'(s)/F(s) = 0$$

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where

$$f(n) := \sum_{\substack{m\in\Lambda \\ m\mid n}} m.$$

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$$\log F(s) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(z) \sum_{n} \frac{f(n)}{n^{z+1}} \frac{\mathrm{d}z}{s^z}$$

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$$\begin{aligned}\log F(s) &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(z)\sum_{n}\frac{f(n)}{n^{z+1}}\frac{\mathrm{d}z}{s^z} \\ &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(z)\zeta(z+1)L(z)\frac{\mathrm{d}z}{s^z}.\end{aligned}$$

Sketch of the rest of the proof

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2. Approximately solve the saddle-point equation, get a good approximation of the saddle point (on the real axis).
3. Use the saddle-point principle to get the contribution of the (neighborhood of) the saddle point to (the integral form) of $p_\Lambda(n)$.
4. Show that the remaining integral is sufficiently small.

Reference

G. Debruyne, G. Tenenbaum, *The saddle-point method for general partition functions*, Indag. Math. 31 (2020), 728-738