18th Balkan Mathematics Olympiad

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(communicated by Dusan Djukic)

1. Let n be a natural number. Show that, if a, b are natural numbers greater than 1 such that $2^n - 1 = ab$, then $ab - (a - b) - 1$ is a number of the form $k2^{2m}$, where k is odd and m natural.

2. In a pentagon all interior angles are congruent and all its sides have rational lengths. Prove that this pentagon is regular.

3. Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq \sqrt{3}abc$.

4. A cube of dimension $3 \times 3 \times 3$ is divided into 27 unit cube cells. One of the cells is empty, and all others are filled with unit cubes which are, on an arbitrary way, denoted with 1,2,...,26. A legal move consists of a move of a unit cube to its neighbouring empty cell. Does there exist a finite sequence of legel moves after which the unit cubes denoted with k and $27 - k$ will exchange their positions for all $k = 1, 2, ..., 13$? (two cells are neighbouring if they have a common face)