19$^{\text{th}}$ Balkan Mathematical Olympiad

Antalya, Turkey

April 27^{th}, 2002

1. Consider n points $A_1, A_2, A_3, ..., A_n$ $(n \geq 4)$ in the plane, such that any three are not collinear. Some pairs of distinct points among $A_1, A_2, A_3, ..., A_n$ are connected by segments, such that every point is connected with at least three different points. Prove that there exists $k > 1$ and the distinct points $X_1, X_2, ..., X_{2k}$ in the set $\{A_1, A_2, A_3, ..., A_n\}$, such that for every $i \in \overline{1, 2k-1}$ the point X_i is connected with X_{i+1}, and X_{2k} is connected with X_1.

2. The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 20, a_2 = 30$ and $a_{n+2} = 3a_{n+1} - a_n$ for every integer $n \geq 1$. Find all positive integers n for which $1 + 5a_na_{n+1}$ is a perfect square.

3. Two circles with different radii intersect in A and B. The common tangents are MN and ST such that M, S lie on the first circle, and N, T on the second. Prove that the orthocenters of $\Delta AMN, \Delta AST, \Delta BMN$ and ΔBST are the vertices of a rectangle.

4. Determine all functions $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that for every positive integer n we have:
$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002.$$

courtesy of Valentin Vornicu