1.3 The final round.

1. Let n be the natural number consisting of 1991 ones:

$$n = \underbrace{111\ldots1}_{1991}$$

Prove that n is not a prime number.

2. (a) Prove that, for all $n \in \mathbb{N}_0$, there exists exactly one positive real number x such that $x^n + x^{n+1} = 1$. Let us call this number x_n.

   (b) Determine $\lim_{n\to+\infty} x_n$.

3. Assume an equilateral triangle abc. Let x denote a point of the edge $]ab[$ (different from a and b). One can now determine in a unique way points y and z such that

   - y lies on $]bc[$
   - z lies on $]ac[$
   - the triangle xyz is equilateral

   Determine the ratios $\frac{|ax|}{|xb|}$, $\frac{|by|}{|yc|}$ and $\frac{|cz|}{|za|}$ if you know that the area of triangle xyz is half the area of triangle abc.

4. A word consisting of n digits chosen between "0" and "1", is called a "bit-string" of length n. (e.g. 000 is a bit-string of length 3 and 01101 is a bit-string of length 5)

   Now, let us consider the sequence $s_1, s_2, \ldots$ of bit-strings of length $n \geq 2$ which we construct as follows:

   (a) s_1 is the bit-string $00\cdots01$ consisting of $n-1$ zeros followed by exactly one "1".

   (b) s_{i+1} is obtained in the following way :

      i. Take s_i and cut the left-most digit. This results in a bit-string t of length $n-1$.

      ii. Look if the bit-string $t1$ is found already in the sequence $\{s_1, \ldots, s_i\}$. If no, take $s_{i+1} = t1$. If yes however, then take $s_{i+1} = t0$.

   As an example we consider this sequence for $n = 3$:

$$s_1 = 001 \to s_2 = 011 \to s_3 = 111 \to s_4 = 110 \to s_5 = 101 \to s_6 = 010 \to$$

$$s_7 = 100 \to s_8 = 000 \to s_9 = 000 \to \cdots$$

   Assume $N = 2^n$. Prove that the bit-strings

$$s_1, s_2, \ldots, s_N$$

   (all of length n) are all different from each other.