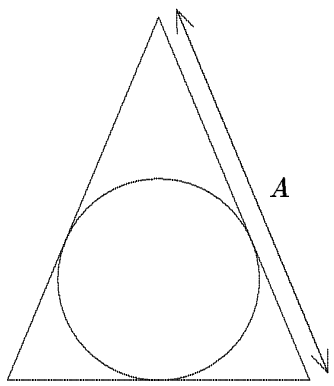
3 The final round.

Problem 1 : Determine, for each positive integer n, the largest positive integer k such that 2^k is a divisor of $3^n + 1$.

Problem 2 : Through an informer from the underworld, the police knows the meeting place of a gang. The identity of the different gang members, however, is unknown. It is the duty of a policeman to shadow the leader of the gang. The policeman knows that this leader is the tallest of the five persons – all of them of different length – which are present at the gang meeting.

After the meeting, the gangsters – as a safety measure – leave the building separately and with an interval time of 15 minutes. As the policeman can not see which one is the tallest, he decides to let the first two gangsters go and to shadow then the first one who is taller then those who left before. Determine the probability that this policeman will shadow the correct person.

Problem 3 : A sphere is inscribed in a cone having apothema equal to A. The tangent circle of the sphere and the cone, determines the upper end of a cylinder which is inscribed in the sphere. Assume that the total area of the cone (mantle as well as base) equals nine times the area of a great circle of the sphere. Assume also that the apothema of the cone is larger than half the perimeter of the base of the cone. Determine the height of the cylinder as a function of A.



Problem 4 : Assume A, B and P are positive numbers such that $P \leq A + B$.

(a) Choose real numbers θ_1 and θ_2 satisfying $A\cos\theta_1 + B\cos\theta_2 = P$ and prove that

$$A\sin\theta_1 + B\sin\theta_2 \leq \sqrt{(A+B-P)(A+B+P)}.$$

(b) Show that the inequality in (a) becomes an equality if $\theta = \theta_1 = \theta_2$ are chosen such that $\cos\theta = \dfrac{P}{A+B}$ with $0 \leq \theta \leq \dfrac{\pi}{2}$.

(c) Take $A = \frac{1}{2}ab$, $B = \frac{1}{2}cd$ and $P = \frac{1}{4}(a^2 + b^2 - c^2 - d^2)$ with $0 < a \leq b \leq a + c + d$, $c, d > 0$ and with $c^2 + d^2 \leq a^2 + b^2$.

Show that the results in (a) and (b) can be interpreted as follows: from all quadrangles having sides with lengths successively a, b, c and d the quadrangle with the largest area is this one for which the vertices lie on a circle.
