3 Flanders Mathematics Olympiad 1996-1997: the Final Round

1. Write 1997 as a sum of natural numbers in such a way that the product of these numbers is as big as possible. Prove that your solution is the best one.

---

2. With respect to an orthonormal basis we consider the curves determined by the equations

$$x^2 + y^2 = r^2 \text{ (with } r \in \mathbb{R}_0^+) \text{ and } (xy)^2 = 1.$$

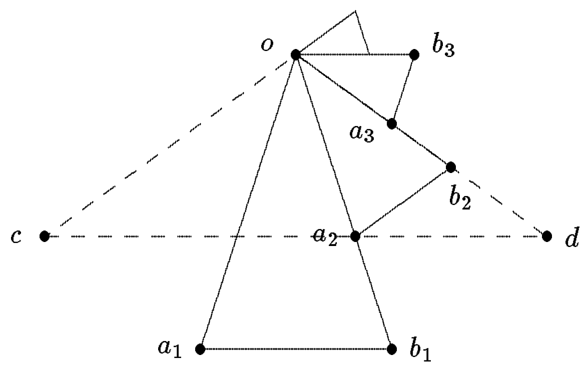
Let us write F_r for the convex polygon having the intersection points of these curves (assuming that there are intersection points) as vertices.

(a) Find an expression $f(r)$ which describes the area of the polygon F_r.

(b) For which values of r is F_r a regular polygon?

---

3. Assume that ℓ is a given length. Consider the isosceles triangle $\Delta_1 = oa_1b_1$, having top-angle $\widehat{a_1ob_1} = 36°$ and two sides of length ℓ.
On the line ob_1 we take a point a_2 such that $|oa_2| = |a_1b_1|$ and we construct - on the outer side of triangle Δ_1 - a second isosceles triangle $\Delta_2 = oa_2b_2$ having top-angle $\widehat{a_2ob_2} = 36°$ (see figure).
We continue working in this direction and constructing a third isosceles triangle $\Delta_3 = oa_3b_3$ with top-angle $\widehat{a_3ob_3} = 36°$ and $|oa_3| = |a_2b_2|$.
Keep repeating this procedure.



Prove that the area of the union of all the triangles, constructed in this way, is smaller than or equal to the area of the isosceles triangle $\Delta = ocd$ with top-angle $\widehat{cod} = 108°$ and two sides $|oc| = |od| = \ell$.

---

4. Consider thirteen birds arriving (and sitting) on a flat surface in such a way that from each 5-tuple of these birds it is known that 4 birds are sitting on a circle.

(a) Prove that there exists a circle such that at least 6 birds are sitting on it.

(b) Can this result be sharpened?
