13 The 1999 Final Round

1. Determine all natural numbers, having 6 digits, say $abcdef$, with $a \neq 0$ and $d \neq 0$ and such that $abcdef = (def)^2$.

2. Let $[mn]$ be a diameter of a circle with a given radius and $[ab]$ a chord with a given (fixed) length on this circle. Assume that $[ab]$ neither coincides with $[mn]$ nor is perpendicular to $[mn]$. Let p be the midpoint of $[ab]$ and let c and d be the orthogonal projections of a and b on $[mn]$. Proof that the angle $\widehat{cpd}$ is independent of the position of the chord $[ab]$.

3. Determine all possible functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ which satisfy the following conditions:

   (1) For all s and $t \in \mathbb{R}$: $2f(s) - g(s) = f(t) - t$;
   (2) For all $s \in \mathbb{R}$: $f(s)\, g(s) \geq s + 1$.

4. Consider a, b, n, $m \in \mathbb{N} \setminus \{0, 1\}$ such that $a^n - 1$ and $b^m + 1$ are prime. Give as much information as possible concerning the integers a, b, n and m.