IRISH MATHEMATICAL OLYMPIAD

2000

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THIRTEENTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 6 May 2000

First Paper

The use of calculators or mathematical tables is not permitted.

Time allowed: Three hours

1. Let S be the set of all numbers of the form $a(n) = n^2 + n + 1$ where n is a natural number. Prove that the product $a(n)a(n+1)$ is in S for all natural numbers n. Give, with proof, an example of a pair of elements $s, t \in S$ such that $st \notin S$.

2. Let $ABCDE$ be a regular pentagon with its sides of length one. Let F be the midpoint of AB and let G, H be points on the sides CD and DE, respectively, such that $\langle GFD = \langle HFD = 30°$. Prove that the triangle GFH is equilateral. A square is inscribed in the triangle GFH with one side of the square along GH. Prove that FG has length

$$t = \frac{2\cos 18°(\cos 36°)^2}{\cos 6°}$$

and that the square has side

$$\frac{t\sqrt{3}}{2+\sqrt{3}}$$

3. Let $f(x) = 5x^{13} + 13x^5 + 9ax$. Find the least positive integer a such that 65 divides $f(x)$ for every integer x.

4. Let

$$a_1 < a_2 < a_3 < \cdots < a_M$$

be real numbers. $\{a_1, a_2, \ldots, a_M\}$ is called a **weak arithmetic progression** of length M if there exist real numbers $x_0, x_1, x_2, \ldots, x_M$ and d such that

$$x_0 \le a_1 < x_1 \le a_2 < x_2 \le a_3 < x_3 \le \cdots \le a_M < x_M$$

and $x_{i+1} - x_i = d$ for $i = 0, 1, 2, \ldots, M-1$, i.e. $\{x_0, x_1, x_2, \ldots, x_M\}$ is an arithmetic progression.

(a) Prove that if $a_1 < a_2 < a_3$ then $\{a_1, a_2, a_3\}$ is a weak arithmetic progression of length 3.

(b) Let A be a subset of $\{0, 1, 2, 3, \ldots, 999\}$ with at least 730 members. Prove that A contains a weak arithmetic progression of length 10.

5. Consider all parabolas of the form $y = x^2 + 2px + q$ (p, q real) which intersect the x- and y-axes in three distinct points. For such a pair p, q let $C_{p,q}$ be the circle through the points of intersection of the parabola $y = x^2 + 2px + q$ with the axes. Prove that all the circles $C_{p,q}$ have a point in common.

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Second Paper

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6. Let $x \geq 0$, $y \geq 0$ be real numbers with $x + y = 2$. Prove that

$$x^2 y^2 (x^2 + y^2) \leq 2.$$

7. Let $ABCD$ be a cyclic quadrilateral and R the radius of the circumcircle. Let a, b, c, d be the lengths of the sides of $ABCD$ and Q its area. Prove that

$$R^2 = \frac{(ab+cd)(ac+bd)(ad+bc)}{16Q^2}$$

Deduce that

$$R \geq \frac{(abcd)^{3/4}}{Q\sqrt{2}}$$

with equality **if and only if** $ABCD$ is a square.

8. For each positive integer n determine with proof, all positive integers m such that there exist positive integers $x_1 < x_2 < \cdots < x_n$ with

$$\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \cdots + \frac{n}{x_n} = m.$$

9. Prove that in each set of ten consecutive integers there is one which is coprime with each of the other integers.
For example, taking $114, 115, 116, 117, 118, 119, 120, 121, 122, 123$ the numbers 119 and 121 are each coprime with all the others. [Two integers a, b are coprime if their greatest common divisor is one.]

10. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with non-negative real coefficients. Suppose that $p(4) = 2$ and that $p(16) = 8$. Prove that $p(8) \leq 4$ and find with proof all such polynomials with $p(8) = 4$.

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