FOURTEENTH IRISH MATHEMATICAL OLYMPIAD
Saturday, 12 May 2001
10a.m.-1p.m.
First Paper

1. Find with proof all solutions of the equation
$$2^n = a! + b! + c!$$
in positive integers a, b, c, n.
(Here ! means "factorial").

2. Let ABC be a triangle with sides BC,CA,AB of lengths a, b, c, respectively. Let D,E be the midpoints of the sides AC, AB, respectively. Prove that the median BD is perpendicular to CE *if and only if*
$$b^2 + c^2 = 5a^2.$$

3. Prove that if an odd prime number p can be expressed in the form $x^5 - y^5$, for some integers x, y , then
$$\sqrt{\frac{4p+1}{5}} = \frac{v^2+1}{2}$$
for some odd integer v.

4. Prove that

   (i)
$$\frac{2n}{3n+1} \leq \sum_{k=n+1}^{2n} \frac{1}{k}$$
   and

   (ii)
$$\sum_{k=n+1}^{2n} \frac{1}{k} \leq \frac{3n+1}{4(n+1)}$$
for all positive integers n.

P.T.O.

5. Let a, b be real numbers such that $ab > 0$. Prove that

$$\sqrt[3]{\frac{a^2b^2(a+b)^2}{4}} \leq \frac{a^2+10ab+b^2}{12}$$

Determine when equality occurs.

Hence, or otherwise, prove for *all* real numbers a, b that

$$\sqrt[3]{\frac{a^2b^2(a+b)^2}{4}} \leq \frac{a^2+ab+b^2}{3}$$

Determine the cases of equality.

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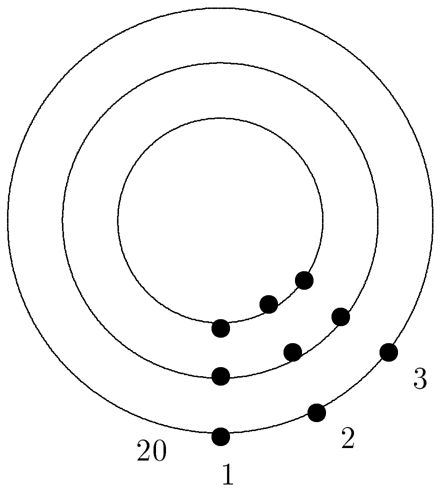
2p.m.-5p.m.

Second Paper

6. Find the least positive integer a such that 2001 divides $55^n + a32^n$ for some odd integer n.

7. Three hoops are arranged concentrically as in the diagram. Each hoop is threaded with 20 beads, of which 10 are black and 10 are white. On each hoop the positions of the beads are labelled 1 through 20 starting at the bottom and travelling counterclockwise.

   We say there is a *match* at position i if all three beads at position i have the same colour. We are free to slide all of the beads around any hoop (but not to unthread and rethread them).



   Show that it is possible (by sliding) to find a configuration involving at least 5 matches.

8. Let ABC be an acute angled triangle, and let D be the point on the line BC for which AD is perpendicular to BC. Let P be a point on the line segment AD. The lines BP and CP intersect AC and AB at E and F respectively. Prove that the line AD bisects the angle EDF.

9. Determine, with proof, all non-negative real numbers x for which

$$\sqrt[3]{13+\sqrt{x}}+\sqrt[3]{13-\sqrt{x}}$$

is an integer.

10. Determine, with proof, all functions f from the set of positive integers to itself which satisfy

$$f(x+f(y))=f(x)+y$$

for all positive integers x, y.