IRISH MATHEMATICAL OLYMPIAD

1994

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An Roinn Oideachais
Forbairt
University of Limerick
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Irish National Mathematics Contest

SEVENTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 7 May 1994

First Paper

The use of calculators or mathematical tables is not permitted.

Time allowed: Three hours

1. Let x, y be positive integers with $y > 3$ and

$$x^2 + y^4 = 2\left[(x-6)^2 + (y+1)^2\right]$$

Prove that $x^2 + y^4 = 1994$.

2. Let A, B, C be three collinear points with B between A and C. Equilateral triangles ABD, BCE, CAF are constructed with D, E on one side of the line AC and F on the opposite side. Prove that the centroids of the triangles are the vertices of an equilateral triangle. Prove that the centroid of this triangle lies on the line AC.

3. Determine with proof all real polynomials $f(x)$ satisfying the equation

$$f(x^2) = f(x)f(x-1).$$

4. Consider the set of $m \times n$ matrices with every entry either 0 or 1. Determine the number of such matrices with the property that the number of "1"s in each row and in each column is even.

5. Let $f(n)$ be defined on the set of positive integers by the rules

$$f(1) = 2 \qquad \text{and} \qquad f(n+1) = \big(f(n)\big)^2 - f(n) + 1; \qquad n = 1, 2, 3, \ldots$$

Prove that for all integers $n > 1$

$$1 - \frac{1}{2^{2^n - 1}} < \frac{1}{f(1)} + \frac{1}{f(2)} + \cdots + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}.$$

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6. A sequence $\{x_n\}$ is defined by the rules

$$x_1 = 2 \qquad \text{and}$$

$$nx_n = 2(2n-1)x_{n-1}; \qquad n = 2, 3, \ldots$$

Prove that x_n is an integer for every positive integer n.

7. Let p, q, r be distinct real numbers which satisfy the equations

$$\begin{aligned} q &= p(4-p) \\ r &= q(4-q) \\ p &= r(4-r). \end{aligned}$$

Find all possible values of $p+q+r$.

8. Prove that for every integer $n > 1$

$$n\left((n+1)^{2/n} - 1\right) < \sum_{i=1}^{n} \frac{2i+1}{i^2} < n\left(1 - n^{-2/(n-1)}\right) + 4.$$

9. Let w, a, b, c be distinct real numbers with the property that there exist real numbers x, y, z for which the following equations hold:

$$\begin{aligned} x \quad + y \quad + z \quad &= 1 \\ xa^2 + yb^2 + zc^2 &= w^2 \\ xa^3 + yb^3 + zc^3 &= w^3 \\ xa^4 + yb^4 + zc^4 &= w^4. \end{aligned}$$

Express w in terms of a, b, c.

10. If a square is partitioned into n convex polygons, determine the maximum number of edges present in the resulting figure.
[You may find it helpful to use a theorem of Euler which states that if a polygon is partitioned into n polygons, then $v - e + n = 1$ where v is the number of vertices and e is the number of edges in the resulting figure].