IRISH MATHEMATICAL OLYMPIAD

1995

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An Roinn Oideachais
University of Limerick
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EIGHTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 6 May 1995

First Paper

The use of calculators or mathematical tables is not permitted.

Time allowed: Three hours

1. There are n^2 students in a class. Each week all the students participate in a table quiz. Their teacher arranges them into n teams of n players each. For as many weeks as possible, this arrangement is done in such a way that any pair of students who were members of the same team one week are not on the same team in subsequent weeks. Prove that after at most $n+2$ weeks, it is necessary for some pair of students to have been members of the same team on at least two different weeks.

2. Determine with proof all those **integers** a for which the equation

$$x^2 + axy + y^2 = 1$$

has infinitely many distinct **integer** solutions x, y.

3. Let A, X, D be points on a line with X between A and D. Let B be a point in the plane such that $\widehat{ABX}$ is greater than 120° and let C be a point on the line between B and X. Prove the inequality

$$2|AD| \geq \sqrt{3}(|AB| + |BC| + |CD|).$$

4. Consider the following one-person game played on the x-axis. For each integer k, let X_k be the point with coordinates $(k, 0)$. During the game disks are piled at some of the points X_k. To perform a move in the game, the player chooses a point X_j at which at least two disks are piled and then takes two disks from the pile at X_j and places one of them at X_{j-1} and one at X_{j+1}.
To begin the game, $2n+1$ disks are placed at X_0. The player then proceeds to perform moves in the game as long as possible. Prove that after $n(n+1)(2n+1)/6$ moves no further moves are possible and that at this stage, one disk remains at each of the positions $X_{-n}, X_{-(n-1)}, \ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots, X_n$.

5. Determine with proof all real-valued functions $f(x)$ satisfying the equation

$$xf(x) - yf(y) = (x-y)f(x+y)$$

for all real numbers x, y.

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6. Prove the inequalities

$$n^n \leq (n!)^2 \leq \left[(n+1)(n+2)/6\right]^n$$

for every positive integer n.

7. Suppose that a, b, c are complex numbers and that all three roots z of the equation

$$x^3 + ax^2 + bx + c = 0$$

satisfy $|z| = 1$ (where $|\quad|$ denotes absolute value). Prove that all three roots w of the equation

$$x^3 + |a|x^2 + |b|x + |c| = 0$$

also satisfy $|w| = 1$.

8. Let S be the square consisting of all points (x, y) in the plane with $0 \leq x, y \leq 1$. For each real number t with $0 < t < 1$, let C_t denote the set of all points $(x, y) \in S$ such that (x, y) is on or above the line joining $(t, 0)$ to $(0, 1-t)$.
Prove that the points common to all C_t are those points in S which are on or above the curve $\sqrt{x} + \sqrt{y} = 1$.

9. We are given three points P, Q, R in the plane. It is known that there is triangle ABC such that P is the midpoint of the side BC, Q is the point on the side CA with $|CQ|/|QA| = 2$ and R is the point on the side AB with $|AR|/|RB| = 2$. Determine with proof how the triangle ABC may be reconstructed from P, Q, R.

10. For each integer n such that $n = p_1\, p_2\, p_3\, p_4$ where p_1, p_2, p_3, p_4 are distinct primes, let

$$d_1 = 1 < d_2 < d_3 < \cdots < d_{15} < d_{16} = n$$

be the sixteen positive integers which divide n.
Prove that if $n < 1995$, then $d_9 - d_8 \neq 22$.