IRISH MATHEMATICAL OLYMPIAD

1996

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An Roinn Oideachais
University of Limerick
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NINTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 4 May 1996

First Paper

The use of calculators or mathematical tables is not permitted.

Time allowed: Three hours

1. For each positive integer n, let $f(n)$ denote the highest common factor of $n! + 1$ and $(n+1)!$ (where ! denotes "factorial"). Find, with proof, a formula for $f(n)$ for each n.

2. For each positive integer n, let $S(n)$ denote the sum of the digits of n (when n is written in base 10). Prove that for every positive integer n

$$S(2n) \leq 2S(n) \leq 10S(2n).$$

Prove also that there exists a positive integer n with

$$S(n) = 1996S(3n).$$

3. Let K be the set of all real numbers x with $0 \leq x \leq 1$. Let f be a function from K to the set of all real numbers $\mathbf{R}$ with the following properties

(i) $f(1) = 1$

(ii) $f(x) \geq 0$ for all $x \in K$

(iii) if x, y and $x + y$ are all in K, then

$$f(x + y) \geq f(x) + f(y).$$

Prove that $f(x) \leq 2x$ for all $x \in K$.

4. Let F be the midpoint of the side BC of the triangle ABC. Isosceles right-angled triangles ABD and ACE are constructed externally on the sides AB and AC with the right angles at D and E, respectively.
Prove that DEF is a right-angled isosceles triangle.

5. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares no two of which have the same area.

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6. The Fibonacci sequences $F_0, F_1, F_2, \ldots$ is defined as follows:
$F_0 = 0$, $F_1 = 1$ and for all $n \geq 0$

$$F_{n+2} = F_n + F_{n+1}$$

(So $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8, \ldots$) Prove that

(i) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers n" is true if $k = 60$ but it is not true for any positive integer $k < 60$

(ii) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers n" is true if $t = 300$ but it is not true for any positive integer $t < 300$.

7. Prove that the inequality

$$2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdots (2^n)^{\frac{1}{2^n}} < 4$$

holds for all positive integers n.

8. Let p be a prime number and a and n positive integers. Prove that if

$$2^p + 3^p = a^n$$

then $n = 1$.

9. Let ABC be an acute-angled triangle and let D, E, F be the feet of the perpendiculars from A, B, C onto the sides BC, CA, AB, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C onto the lines EF, FD, DE respectively. Prove that the lines AP, BQ, CR (extended) are concurrent.

10. We are given a rectangular chessboard of size 5×9 (so there are five rows of squares, each row containing nine squares). The following game is played:
Initially, a number of discs are randomly placed on some of the squares, no square being allowed to contain more than one disc. A complete move consists of moving every disc from the square containing it to another square subject to the following rules

(i) each disc may be moved one square up or down, or left or right, of the square it occupies, to an adjoining square;

(ii) if a particular disc is moved up or down as part of a complete move, then it must be moved left or right in the next complete move;

(iii) if a particular disc is moved left or right as part of a complete move, then it must be moved up or down in the next complete move;

(iv) at the end of each complete move, no square can contain two or more discs.

The game stops if it becomes impossible to perform a complete move. Prove that if initially 33 discs are placed on the board then the game must eventually stop.
Prove also that it is possible to place 32 discs on the board in such a way that the game could go on forever.

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