IRISH MATHEMATICAL OLYMPIAD

1997

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An Roinn Oideachais
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IMTA

TENTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 10 May 1997

First Paper

The use of calculators or mathematical tables is not permitted.

Time allowed: Three hours

1. Find (with proof) all pairs of integers (x, y) satisfying the equation

$$1 + 1996x + 1998y = xy.$$

2. Let ABC be an equilateral triangle.
For a point M inside ABC, let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right angle.

3. Find all polynomials $p(x)$ satisfying the equation

$$(x - 16)p(2x) = 16(x - 1)p(x)$$

for all x.

4. Let a, b, c be nonnegative real numbers. Suppose that $a + b + c \geq abc$. Prove that

$$a^2 + b^2 + c^2 \geq abc$$

5. Let S be the set of all odd integers greater than one. For each $x \in S$, denote by $\delta(x)$ the unique integer satisfying the inequality

$$2^{\delta(x)} < x < 2^{\delta(x)+1}.$$

For $a, b \in S$, define

$$a * b = 2^{\delta(a)-1}(b - 3) + a.$$

[For example, to calculate $5 * 7$ note that $2^2 < 5 < 2^3$, so $\delta(5) = 2$, and hence $5 * 7 = 2^{2-1}(7 - 3) + 5 = 13$. Also $2^2 < 7 < 2^3$, so $\delta(7) = 2$ and $7 * 5 = 2^{2-1}(5 - 3) + 7 = 11$].
Prove that if $a, b, c \in S$, then
(i) $a * b \in S$ and
(ii) $(a * b) * c = a * (b * c)$.

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6. Given a positive integer n, denote by $\sigma(n)$ the sum of all the positive integers which divide n. [For example, $\sigma(3) = 1 + 3 = 4$, $\sigma(6) = 1 + 2 + 3 + 6 = 12$, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$].
We say that n is *abundant* if $\sigma(n) > 2n$. (So, for example, 12 is abundant).
Let a, b be positive integers and suppose that a is abundant. Prove that ab is abundant.

7. $ABCD$ is a quadrilateral which is circumscribed about a circle Γ (i.e. each side of the quadrilateral is tangent to Γ). If $\angle A = \angle B = 120°$, $\angle D = 90°$ and BC has length 1, find, with proof, the length of AD.

8. Let A be a subset of $\{0, 1, 2, 3, \ldots, 1997\}$ containing more than 1000 elements. Prove that either A contains a power of 2 (that is, a number of the form 2^k with k a nonnegative integer) or there exist two distinct elements $a, b \in A$ such that $a + b$ is a power of 2.

9. Let S be the set of all natural numbers n satisfying the following conditions:
(1) n has 1000 digits
(2) all the digits of n are odd and
(3) the absolute value of the difference between adjacent digits of n is 2.
Determine the number of distinct elements in S.

10. Let p be an odd prime number and n a natural number and let $T = \{1, 2, 3, \ldots, n\}$. Then n is called *p-partitionable* if there exist nonempty subsets $T_1, T_2, \ldots, T_p$ of T such that
(i) $T = T_1 \cup T_2 \cup \cdots \cup T_p$
(ii) $T_1, T_2, \ldots, T_p$ are disjoint (that is $T_i \cap T_j$ is the empty set for all i, j with $i \neq j$) and
(iii) the sum of the elements in T_i is the same for $i = 1, 2, \ldots, p$.
[For example, 5 is 3-partitionable since, if we take $T_1 = \{1, 4\}$, $T_2 = \{2, 3\}$ $T_3 = \{5\}$, then (i), (ii) and (iii) are satisfied. Also 6 is 3-partitionable since, if we take $T_1 = \{1, 6\}$, $T_2 = \{2, 5\}$, $T_3 = \{3, 4\}$, then (i), (ii) and (iii) are satisfied].
(a) Suppose that n is p-partitionable. Prove that p divides n or $n + 1$.
(b) Suppose that n is divisible by $2p$. Prove that n is p-partitionable.