IRISH MATHEMATICAL OLYMPIAD

1998

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IMTA

ELEVENTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 9 May 1998

First Paper

The use of calculators or mathematical tables is not permitted.

Time allowed: Three hours

1. Show that if x is a nonzero real number, then

$$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0.$$

2. P is a point inside an equilateral triangle such that the distances from P to the three vertices are 3, 4 and 5, respectively. Find the area of the triangle.

3. Show that no integer of the form $xyxy$ in base 10 (where x and y are digits) can be the cube of an integer.
Find the smallest base $b > 1$ for which there is a perfect cube of the form $xyxy$ in base b.

4. Show that a disc of radius 2 can be covered by seven (possibly overlapping) discs of radius 1.

5. If x is a real number such that $x^2 - x$ is an integer, and, for some $n \geq 3$, $x^n - x$ is also an integer, prove that x is an integer.

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Second Paper

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Time allowed: Three hours

6. Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2, \ldots, d_{16}$ such that

$$1 = d_1 < d_2 < \cdots < d_{16} = n,$$

$d_6 = 18$ and $d_9 - d_8 = 17$.

7. Prove that if a, b, c are positive real numbers, then

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right), \tag{1}$$

and

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \tag{2}$$

8. Let N be the set of natural numbers (i.e., the positive integers).

(a) Prove that N can be written as a union of three mutually disjoint sets such that, if $m, n \in N$ and $|m-n| = 2$ or 5, then m and n are in different sets.

(b) Prove that N can be written as a union of four mutually disjoint sets such that, if $m, n \in N$ and $|m-n| = 2, 3$ or 5, then m and n are in different sets. Show, however, that it is impossible to write N as a union of three mutually disjoint sets with this property.

9. A sequence of real numbers x_n is defined recursively as follows: x_0, x_1 are arbitrary positive real numbers, and

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}, \ n = 0, 1, 2, \ldots$$

Find x_{1998}.

10. A triangle ABC has positive integer sides, $\widehat{A} = 2\widehat{B}$ and $\widehat{C} > 90°$. Find the minimum length of the perimeter.

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