IRISH MATHEMATICAL OLYMPIAD

1999

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IMTA

TWELFTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 8 May 1999

First Paper

The use of calculators or mathematical tables is not permitted.

Time allowed: Three hours

1. Find all real values of x which satisfy

$$\frac{x^2}{(x+1-\sqrt{x+1})^2} < \frac{x^2+3x+18}{(x+1)^2}.$$

2. Show that there is a positive number in the Fibonacci sequence which is divisible by 1000.
[The Fibonacci sequence F_n is defined by the conditions $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. So the sequence begins $0, 1, 1, 2, 3, 5, 8, 13, \ldots$]

3. Let D, E, F be points on the sides BC, CA, AB, respectively, of triangle ABC such that AD is perpendicular to BC, BE is the angle-bisector of $\angle B$ and F is the midpoint of AB. Prove that AD, BE, CF are concurrent if and only if

$$a^2(a-c) = (b^2 - c^2)(a+c),$$

where a, b, c are the lengths of the sides BC, CA, AB, respectively, of triangle ABC.

4. A square floor consisting of 10000 squares (100 squares $\times$ 100 squares - like a large chessboard) is to be tiled. The only available tiles are rectangular 1×3 tiles, fitting exactly over three squares of the floor.
(i) If a 2×2 square is removed from the centre of the floor, prove that the remaining part of the floor can be tiled with the available tiles.
(ii) If, instead, a 2×2 square is removed from the corner, prove that the remaining part of the floor cannot be tiled with the available tiles.
[There are sufficiently many tiles available. To *tile* the floor - or a portion thereof - means to completely cover it with the tiles, each tile covering three squares, and no pair of tiles overlapping. The tiles may not be broken or cut.]

5. Three numbers $a < b < c$ are said to be in arithmetic progression if $c - b = b - a$.
Define a sequence u_n, $n = 0, 1, 2, 3, \ldots$ as follows:

$$u_0 = 0, \qquad u_1 = 1$$

and for each $n \geq 1$, u_{n+1} is the smallest positive integer such that $u_{n+1} > u_n$ and $\{u_0, u_1, \ldots, u_n, u_{n+1}\}$ contains no three elements which are in arithmetic progression.
Find u_{100}.

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Second Paper

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6. Solve the system of (simultaneous) equations

$$y^2 = (x+8)(x^2+2),$$

$$y^2 - (8+4x)y + (16+16x-5x^2) = 0.$$

7. A function $f : \mathbf{N} \to \mathbf{N}$ (where $\mathbf{N}$ denotes the set of positive integers) satisfies
(1) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1,
(2) $f(p+q) = f(p) + f(q)$ for all prime numbers p and q.
Prove that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

8. Let a, b, c and d be positive real numbers whose sum is 1. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2},$$

with equality if and only if $a = b = c = d = 1/4$.

9. Find all positive integers m with the property that the fourth power of the number of (positive) divisors of m equals m.

10. $ABCDEF$ is a convex (not necessarily regular) hexagon with $AB = BC$, $CD = DE$, $EF = FA$ and

$$\angle ABC + \angle CDE + \angle EFA = 360^\circ.$$

Prove that the perpendiculars from A, C and E to FB, BD and DF respectively are concurrent.

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