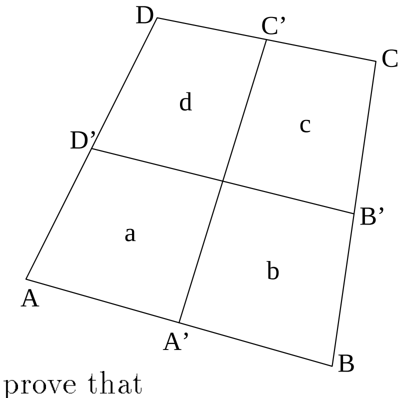
The Niels Henrik Abel Contest 1993
FINAL

Problem 1



a) Let $ABCD$ be a convex quadrilateral. (Ie. the angles are all less than $180°$.) Let A' be the midpoint of AB, B' the midpoint of BC, C' the midpoint of CD, and D' the midpoint of AD. Draw the lines $A'C'$ and $B'D'$, and let a, b, c, and d be the areas of the four minor quadrilaterals, as shown in the figure. Prove that $a + c = b + d$.

b) Given a triangle with sides of length a, b, and c, prove that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} < 2.$$

Problem 2

Prove that if $b < c < d$, then the inequality

$$(a + b + c + d)^2 > 8(ac + bd)$$

holds for all a.

Problem 3

The Fermat-numbers are defined by $F_n = 2^{2^n} + 1$ for $n = 0, 1, 2, \ldots$

a) Prove that $F_n = F_{n-1}F_{n-2} \cdots F_1F_0 + 2$ for $n = 1, 2, 3, \ldots$

b) Prove that two different Fermat-numbers cannot have a common factor greater than 1.

Problem 4

We have a cube. Each of the 8 corners are given values 1 or -1. Each of the six sides are given a value equaling the product of its four corners. Let A be the sum of the values of all the corners (eight) and all the sides (six). Then, A is the sum of 14 values. Which values of A are attainable?