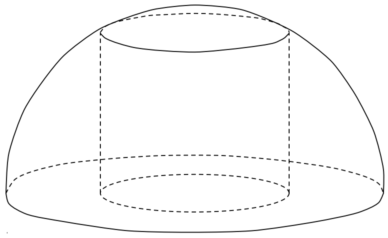
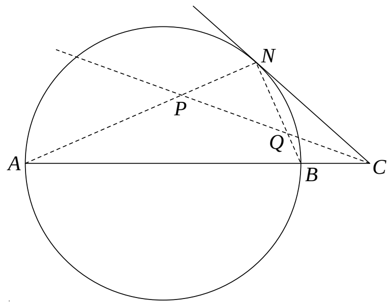
The Niels Henrik Abel Contest 1994
FINAL

Problem 1



a) We have one half of a ball of radius 3. Inside this is a cylinder of radius $\sqrt{3}$ placed in the middle of the bottom plane and which touches the surface of the ball. There is an other such cylinder with a different radius that has the same volume. What radius does this have?



b) Let AB be the diameter of a circle, and let C be a point on the prolongation of AB. Draw a line through C which touches the circle in the point N. The birector of the angle $\angle ACN$ intersects the lines AN and BN in the points P and Q. Prove that $PN = QN$.

Problem 2

a) Find all primes p, q, r and natural numbers $n \in \mathbf{N}$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{n}$.

b) Find all integers $x, y, z \in \mathbf{Z}$ such that $x^3 + 5y^3 = 9z^3$.

Problem 3

a) Let $x_1, x_2, \ldots, x_{1994} > 0$ be real numbers. Prove that

$$\left(\frac{x_1}{x_2}\right)^{\frac{x_1}{x_2}} \left(\frac{x_2}{x_3}\right)^{\frac{x_2}{x_3}} \cdots \left(\frac{x_{1993}}{x_{1994}}\right)^{\frac{x_{1993}}{x_{1994}}} \geq \left(\frac{x_1}{x_2}\right)^{\frac{x_2}{x_1}} \left(\frac{x_2}{x_3}\right)^{\frac{x_3}{x_2}} \cdots \left(\frac{x_{1993}}{x_{1994}}\right)^{\frac{x_{1994}}{x_{1993}}}.$$

b) Prove that there exists no function $f : \mathbf{Z} \to \mathbf{Z}$ (both arguments and results are integers) such that $f(f(x)) = x + 1$ for all x.

Problem 4

a) In a group of 20 people, at some point of time each person sends a letter to 10 of the others. Prove that there are two persons who each send a letter to the other.

b) A finite number of cities are connected by one-way roads. For each pair of cities, there are at least one path (possibly through other cities) from one to the other, but not necessarily both ways. Prove that there is a city which may be reached from all the other cities, and that there is a city from which all other cities may be reached.