Telenor

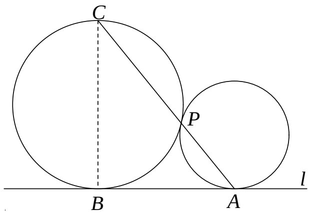
The Niels Henrik Abel Contest 1994–95
FINAL

Problem 1

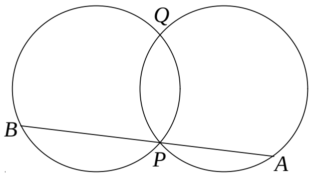
a) Let $f(1) = 1$, $f(1)+f(2)+f(3)+\cdots+f(n) = n^2 \cdot f(n)$ for all natural numbers n. What is $f(1995)$?

b) Prove that if $(x+\sqrt{x^2+1})(y+\sqrt{y^2+1}) = 1$, then $x+y=0$.

Problem 2



a) Two circles touch a line l in the points A and B, and each other in a point P. The line AP intersects the other circle in C. Prove that BC is normal to l.



b) Two circles with the same radii intersect in two different points: P and Q. Draw a line through P which is not tangential to any of the circles. In addition to P, this line intersects the circles in points A and B. Prove that the perpendicular bisector of AB passes through Q.

Problem 3

Prove that there exists an ordering of the natural numbers, ie. a sequence x_i where $i = 1,2,3,4,\ldots$ such that every natural number occurs exactly once, and so that the sums $\sum_{i=1}^{n} 1/x_i$ for $n = 1,2,3,4,\ldots$ include all natural numbers. Ie. for every natural number m, there exists an n so that $m = \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\cdots+\frac{1}{x_n}$.

Problem 4

Let n be a natural number, and let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n > 0$. Prove that

$$\left(\sum_{i=1}^{n}(x_i+y_i)^2\right)\cdot\left(\sum_{i=1}^{n}\frac{1}{x_i y_i}\right) \geq 4n^2.$$

Ie. that $((x_1+y_1)^2+\cdots+(x_n+y_n)^2)\cdot\left(\frac{1}{x_1y_1}+\cdots+\frac{1}{x_ny_n}\right) \geq 4n^2$.