12th Nordic Mathematical Contest

Thursday April 2nd, 1998

English version

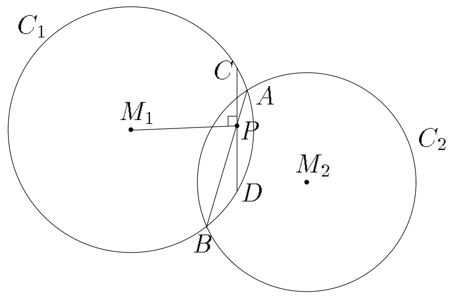
Time allowed: 4 hours. A maximum of 5 points scored per problem.

Problem 1

Find all functions f from the rational numbers to the rational numbers satisfying $f(x+y)+f(x-y)=2f(x)+2f(y)$ for all rational x and y.

Problem 2

Let C_1 and C_2 be two circles which intersect at points A and B. Let M_1 be the centre of C_1 and M_2 the centre of C_2. Let P be a point on the linesegment AB distinct from A and B and so that $|AP| \neq |BP|$. Draw the line through P perpendicular to M_1P and denote by C and D its intersections with C_1 (see figure). Similarly (not drawn in the figure), draw the line through P perpendicular to M_2P and denote by E and F its intersections with C_2. Prove that C, D, E, and F are the corners of a rectangle.



Problem 3

a) For which positive integers n does there exist a sequence $x_1, x_2, \ldots, x_n$ containing each of the numbers $1, 2, \ldots, n$ exactly once and such that k divides $x_1 + x_2 + \cdots + x_k$ for $k = 1, 2, \ldots, n$?

b) Does there exist an infinite sequence $x_1, x_2, x_3, \ldots$ containing every positive integer exactly once and such that for any positive integer k, k divides $x_1 + x_2 + \cdots + x_k$?

Problem 4

Let n be a positive integer. Count the number of $k \in \{0, 1, 2, \ldots, n\}$ for which $\binom{n}{k}$ is odd. Prove that this number is a power of two: ie., on the form 2^p for some non-negative integer p.