51-st Mathematical Olympiad in Poland

Final Round, April 3–4, 2000

First Day

1. Let $n \geq 2$ be a given integer. How many solutions $(x_1, x_2, \ldots, x_n)$ possesses the following system of equations

$$\begin{cases} x_2 + x_1^2 = 4x_1 \\ x_3 + x_2^2 = 4x_2 \\ x_4 + x_3^2 = 4x_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_n + x_{n-1}^2 = 4x_{n-1} \\ x_1 + x_n^2 = 4x_n \end{cases}$$

in the set of nonnegative real numbers?

2. In the triangle ABC holds $AC = BC$. The point P lies inside the triangle ABC and is chosen such that $\angle PAB = \angle PBC$. The point M is the midpoint of the side AB. Prove that

$$\angle APM + \angle BPC = 180^\circ.$$

3. The sequence (p_n) of natural numbers satisfies:
1° p_1 and p_2 are primes,
2° for $n \geq 3$ the number p_n is the greatest divisor of the number

$$p_{n-1} + p_{n-2} + 2000\ .$$

Prove that the sequence (p_n) is bounded.

Second Day

4. In the regular pyramid with the vertex S and the base $A_1A_2...A_n$ each lateral edge makes the angle 60° with the base of the pyramid. For each natural number $n \geq 3$ prove or disprove that:
there exist points $B_2, B_3, \ldots, B_n$ lying on the edges $A_2S, A_3S, \ldots, A_nS$, respectively such that the following inequality holds

$$A_1B_2 + B_2B_3 + B_3B_4 + \ldots + B_{n-1}B_n + B_nA_1 < 2A_1S\,.$$

5. For the given natural number $n \geq 2$ find the smallest number k with the following property: From each set of k unit squares of the $n \times n$ chessboard one can choose a subset such that the number of the unit squares contained in this subset and lying in a row or column of the chessboard is even.

6. Let $P(x)$ be a real polynomial with an odd degree satisfying:

$$P(x^2 - 1) = (P(x))^2 - 1$$

for all x. Prove that for all x the following equality holds

$$P(x) = x\,.$$