52-nd Mathematical Olympiad in Poland

Final Round, April 3–4, 2001

First Day

1. Prove that for each positive integer $n \geq 2$ and for all nonnegative real numbers $x_1, x_2, \ldots, x_n$ the following inequality holds

$$\sum_{i=1}^{n} i x_i \leq \binom{n}{2} + \sum_{i=1}^{n} x_i^i \,.$$

2. Prove that for any interior point P of a regular tetrahedron with edge equal to 1 the following statement holds:
the sum of the distances from P to the vertices of the tetrahedron is not greater than 3.

3. Consider the sequence $\{x_n\}$ defined by

$$x_1 = a, \quad x_2 = b, \quad x_{n+2} = x_{n+1} + x_n \text{ for } n = 1, 2, 3, \ldots$$

where a, b are real numbers. We will call a number c a *multiple value* of the sequence $\{x_n\}$ if there exist positive integers $k \neq l$ such that $x_k = x_l = c$. Prove that there exist the values a, b such that $\{x_n\}$ possesses more than 2000 different multiple values. Moreover prove that $\{x_n\}$ cannot possess infinitely many different multiple values.

Second Day

4. Let a, b be integers such that for all positive integers n the number $2^n a + b$ is the square of an integer. Prove that $a = 0$.

5. On the sides BC and CD of the parallelogram $ABCD$ lie the points K and L, respectively and are so chosen that $BK \cdot AD = DL \cdot AB$. The segments DK and BL intersect in the point P. Prove that $\sphericalangle DAP = \sphericalangle BAC$.

6. Let $n_1 < n_2 < \ldots < n_{2000} < 10^{100}$ be given positive integers. Prove that the set $\{n_1, n_2, \ldots, n_{2000}\}$ contains two non empty and disjoint subsets A, B such that:
A and B have the same number of elements,
the sums of the elements of A and of B are equal,
the sums of the squares of the elements of A and of B are equal.