53rd Mathematical Olympiad in Poland

Problems of the first round, September – December 2001

1. Solve the following equation in real numbers

$$|x| - |x+2| + |x+4| - |x+6| + \ldots - |x+998| =$$
$$= |x+1| - |x+3| + |x+5| - |x+7| + \ldots - |x+999|$$

2. ABC is a given triangle. $ABDE$ and $ACFG$ are the squares drawn outside of the triangle. The points M and N are the midpoints of DG and EF respectively. Find all values of the ratio $MN : BC$.

3. Prove that the number

$$\sum_{n=0}^{10^{10}} \binom{2 \cdot 10^{10}}{2n} 5^n$$

is divisible by $2^{2 \cdot 10^{10} - 1}$.

4. Prove that the graph of the polynomial $W(x)$ with $\deg \mathrm{W} > 1$ possesses a symmetry axis if and only if there exist polynomials $F(x), G(x)$ such that $W(x) = F(G(x))$ and $\deg \mathrm{G} = 2$.

5. Prove that for each positive integer k there exists a positive integer m such that all the numbers $m, 2m, 3m, \ldots, m^2$ have exactly k nonzero digits in the binomial expansion.

6. A circle divides all sides of a rhombus into three pieces. Starting from a vertex of the rhombus and going in a fixed direction along the boundary of the rhombus the 12 segments are coloured red, green and white successively. Prove that the sum of the lengths of the red segments is equal to the sum of the lengths of the white segments.

7. In a group of $n \geq 3$ peoples each member of the group has an even number (perhaps zero) of acquaintances in the group. Prove that there exist three members of the group which have the same number of acquaintances in this group.

Remark: Assume that nobody includes himself into the set of his acquaintances and that A knows B if and only if B knows A.

8. Let $S(n)$ denotes the sum of digits of the number n. Prove that for each positive integer n the number $S(2n^2 + 3)$ is not the square of an integer.

9. A plain intersects lateral edges of a prism with a hexagonal base in the points $D_1, D_2, \ldots, D_6$. The intersection set $D_1D_2D_3D_4D_5D_6$ is a convex hexagon. Denote by d_i the distance of the point D_i to the plain containing one fixed base of the prism. Prove that $d_1^2 + d_3^2 + d_5^2 = d_2^2 + d_4^2 + d_6^2$.

10. On each field of a 2000×2000 chessboard lies a stone. The stones can be moved in the following manner: Take three successive fields lying in a row or in a column. If on the first and on the third field lies a stone then these two stones can be moved on the second field. (Note that a movement can be executed independent of the number of stones lying on the middle field.) Prove or disprove: There exists a sequence of movements such that at the end all the stones lie on one field of the chessboard.

11. In the triangle ABC it holds that $\sphericalangle B > \sphericalangle C$. The point D lies on the side BC and satisfies the equality $\sphericalangle DAC = (1/2)(\sphericalangle B - \sphericalangle C)$. The circle tangent to the line AC in the point A and containing the point D intersects the line AB in the point $P \neq A$. Prove that

$$\frac{BP}{AC} = \frac{BD}{DC}\,.$$

12. In a non-decreasing sequence $a_1, a_2, a_3, \ldots$ all values are positive integers and exactly k values are equal to k. Find all primes of the form

$$a_1 + a_2 + \ldots + a_n\,.$$